Published by Institute of Physics Publishing for SISSA
Received: *August 23, 2005*
Accepted: *November 5, 2005*
Published: *November 18, 2005*

On the non-commutative geometry of topological D-branes

Calin Iuliu Lazaroiu
Department of Mathematics, Trinity College Dublin
Dublin 2, Ireland
E-mail: calin@maths.tcd.ie

Abstract: This is a noncommutative-geometric study of the semiclassical dynamics of finite topological D-brane systems. Starting from the formulation in terms of A_∞ categories, I show that such systems can be described by the noncommutative symplectic supergeometry of $\mathbb{Z}_2$-graded quivers, and give a synthetic formulation of the boundary part of the generalized WDVV equations. In particular, a *faithful* generating function for integrated correlators on the disk can be constructed as a linear combination of quiver necklaces, i.e. a function on the noncommutative symplectic superspace defined by the quiver's path algebra. This point of view allows one to construct extended moduli spaces of topological D-brane systems as non-commutative algebraic 'superschemes'. They arise by imposing further relations on a $\mathbb{Z}_2$-graded version of the quiver's preprojective algebra, and passing to the subalgebra preserved by a natural group of symmetries.

Keywords: Topological Strings, Non-Commutative Geometry, D-branes.

Contents

1. Introduction

The extended moduli spaces [1] of closed topological strings are Frobenius supermanifolds [2], certain types of flat riemannian supermanifolds whose metric is induced by the generating functional of tree-level closed string amplitudes. This description follows from the consistency constraints on closed string amplitudes on the sphere, known as the WDVV equations [3]. The theory of Frobenius supermanifolds encodes many interesting properties such as the generic lack of obstructions[1] of (not necessarily conformal) topological bulk deformations, and has well-known applications to closed string mirror symmetry.

It is natural to ask if a similar description can be given for topological deformations of open strings. A boundary topological string theory admits two types of deformations, which are induced by bulk and boundary observables. As in the closed string case, the open-closed tree-level amplitudes obey consistency conditions (the so-called *generalized WDVV equations*), which were derived in [7] by worldsheet arguments. While bulk deformations have the same character as in the boundary-free case, boundary deformations behave quite differently. As shown in [7], they are constrained by a *homotopy* version of the associativity conditions, leading to an intricate structure known as a cyclic and unital weak A_∞ category (see [8 − 13, 41 − 46] for related work). The complication arises due to the non-commutativity of boundary insertions on the disk, and leads to difficulties when analyzing the boundary moduli problem. Among these is the observation that the the homotopy associativity constraints for the integrated boundary disk amplitudes $W_{a_1 \ldots a_n}$ cannot be 'integrated' faithfully to an ordinary generating function of the boundary deformation parameters.

In the present paper, we investigate a resolution to this problem, by arguing quite generally that the semiclassical dynamics of open strings in finite topological D-brane systems can be described in the framework of supersymplectic noncommutative geometry. This approach, which is already implicit in the A_∞ constraints of [7], leads us to consider

[1] A manifestation of this is the Bogomolov-Tian-Todorov theorem [5, 6] on unobstructedness of deformations of complex structure for Calabi-Yau manifolds, and its generalization to extended deformations due to Barannikov and Kontsevich [4]. The extended moduli space of complex structures for such manifolds coincides with the moduli space of deformations of the associated topological B-type string.

the boundary deformation potential as a function on a *noncommutative* space, and allows for a synthetic formulation of the boundary WDVV equations. Moreover, it leads naturally to a construction of boundary moduli spaces of topological D-branes as noncommutative superspaces.

Let us explain this for the simple example of a single topological D-brane. In this situation, one could try the following naive proposal for the generating function:

$$W_{naive} = \sum_n \frac{1}{n} W_{a_1 \ldots a_n} \sigma^{a_1} \ldots \sigma^{a_n} ,$$

where one views the boundary deformation parameters as (super)commuting variables σ^a. However, supercommutativity of the parameters implies that W_{naive} reduces to:

$$\sum_n \frac{1}{n} W_{(a_1 \ldots a_n)} \sigma^{a_1} \ldots \sigma^{a_n} ,$$

where $W_{(a_1 \ldots a_n)}$ is the (super) symmetrization of the amplitudes. Thus a generating function based on (super) commuting deformation parameters does not faithfully encode the topological tree-level data of the worldsheet theory, and cannot generally be used to reconstruct the latter.

It was suggested in [7] (see also [12]) that this problem might be overcome by viewing the boundary deformation parameters as *non-commuting.* While this might seem unusual at first sight, it is in fact quite natural if one recalls that any boundary theory admits Chan-Paton extensions, whose effect is to promote the deformation parameters to (super)matrices X^a. As a result, some of the information lost by W_{naive} is preserved by the matrix potential:

$$\hat{W} = \sum_n \frac{1}{n} W_{a_1 \ldots a_n} \mathrm{str}(X^{a_1} \ldots X^{a_n}) .$$

However, supertraces of matrix monomials of finite dimensions generally obey polynomial constraints. As a consequence, the matrix potential of a fixed Chan-Paton extension can be reduced by such relations, and again fails to faithfully encode the data of the theory. To completely resolve the issue, one has to remove all constraints on X^a, which amounts to replacing them by free (and in particular non-commuting) supervariables s^a. Hence one is lead to the *non-commutative generating function* of [7]:

$$W = \sum_n \frac{1}{n} W_{a_1 \ldots a_n} (s^{a_1} \ldots s^{a_n})_c ,$$

where $(.)_c$ denotes the graded-cyclization operation, which gives an abstract analogue of the supertrace. Notice that W allows one to recover any Chan-Paton extension upon replacing s^a with supermatrices X^a, which amounts to considering a finite-dimensional representation of the free associative superalgebra $A = \mathbb{C}\langle\{s^a\}\rangle$. In this way, one can study at once *all* Chan Paton extensions of the theory, as well as more general representations obtained by taking morphisms from A to an arbitrary associative superalgebra.

JHEP11(2005)032

Remarkably, the procedure outlined above agrees with a key principle of noncommutative algebraic geometry espoused in [14] and developed further in [15] (see [16] for an introduction). According to this ideology, 'good' notions in affine non-commutative algebraic geometry should induce the corresponding classical notions on the moduli spaces of finite-dimensional representations of the noncommutative coordinate ring. In the example above, the free superalgebra A is the coordinate ring of a noncommutative affine superspace $\mathbb{A}$, while finite-dimensional representations of this algebra (i.e. supermatrix-valued points of the 'noncommutative scheme' $\mathbb{A}$) correspond to Chan-Paton extensions of the theory. By insisting that the generating function should faithfully encode the information of integrated amplitudes on the disk, we are lead to consider W as an element of the cyclic subspace A_c of A (namely the subspace of A spanned by all graded-cyclic monomials in the generators s^a). This matches the interpretation [14] of A_c as the space of regular functions on $\mathbb{A}$. Thus W is a function on a non-commutative affine space, and we find that physics reasoning agrees with the approach to non-commutative algebraic geometry advocated in [14, 15]. Moreover, one can show that the boundary topological metric induces an (even or odd) noncommutative symplectic form on $\mathbb{A}$, which makes this affine superspace into a noncommutative symplectic supermanifold. As in the supercommutative case, the symplectic structure determines a bracket $\{.,.\}$ on A_c, and one finds that the homotopy associativity constraints of [7] are equivalent with the equation:

$$\{W, W\} = 0\,. \tag{1.1}$$

Moreover, the unitality condition [7, 8] on the underlying A_∞ algebra can also be formulated as a constraint on W. This gives a non-commutative geometric interpretation of the boundary part of the generalized WDVV equations. As explained in [7], the boundary topological metric of a general worldsheet theory can be even or odd; as a consequence, $\{.,.\}$ is an even or odd Lie bracket. In the latter case, the constraint (1.1) is a *non-commutative* analogue of the classical master equation.

Non-commutativity of boundary observable insertions is also responsible for the fact that deformations of topological D-branes are generally obstructed, which is reflected in the typically singular nature of the boundary moduli space. An algebro-geometric approach to boundary deformations was developed in relation to homological mirror symmetry [17] in [41] and related to the deformation theory of open strings in [43] (see [18] for related work); these proposals rely on constructing the moduli space as a commutative algebraic or complex-analytic variety.

The observations made above suggest that the deformation theory of topological D-branes can be considered as a problem in *noncommutative* geometry. In particular, the possibility of Chan-Paton extensions implies that the boundary moduli space can be viewed as a non-commutative algebraic variety. This is obtained by 'extremizing' the noncommutative function W and modding out via appropriate symmetries. More precisely, one can impose the algebraic relations:

$$\overrightarrow{\delta}_a W = 0\,, \tag{1.2}$$

where $\overrightarrow{\delta}_a$ is a $\mathbb{Z}_2$-graded version of the so-called *cyclic derivatives* of [19, 20] (see also [14]). If J is the two-sided ideal generated by $\overrightarrow{\delta}_a W$, then the quotient algebra $\mathbb{C}[\mathcal{Z}] := \mathbb{C}[\{s^a\}]/J$ can be viewed as the coordinate ring of a 'noncommutative affine scheme' $\mathcal{Z}$ sitting inside the affine superspace $\mathbb{A}$. Moreover, one can show that the symmetries of the system form a subgroup $\mathcal{G}$ of the group of noncommutative symplectomorphisms of $\mathbb{A}$. These symmetries also preserve J, and thus descend to automorphisms of $\mathbb{C}[\mathcal{Z}]$. They can be viewed as gauge transformations acting along the 'noncommutative vacuum space' $\mathcal{Z}$. One can thus define a *non-commutative extended moduli space* $\mathcal{M}$ as the affine 'noncommutative scheme' whose coordinate ring $\mathbb{C}[\mathcal{M}] = \mathbb{C}[\mathcal{Z}]^{\mathcal{G}}$ is the $\mathcal{G}$-invariant part of $\mathbb{C}[\mathcal{Z}]$. The existence of a unit observable in the boundary sector implies that one of the conditions (1.2) is a non-commutative moment map constraint in the sense of [21, 22]. Therefore, the (invariant theory) quotient leading to $\mathcal{M}$ amounts to modding out a zero-level 'symplectic reduction' of A through the ideal defined by the remaining relations.

It turns out that the construction outlined above can be carried out in much greater generality. In fact, as pointed out in [8, 7], the homotopy-associativity constraints on disk boundary amplitudes generalize to *systems* of D-branes. In this situation, boundary observables are either boundary-preserving or boundary condition-changing, a decomposition which defines the boundary sectors discussed in [23]. The homotopy associativity constraints of [7] admit an obvious extension to this case, which can be formulated by saying [8] that the D-brane system defines a (generally weak) cyclic and unital A_∞ category. The objects u of this category are the D-branes themselves, while the morphism spaces $\mathrm{Hom}(u, v) = E_{uv}$ are the spaces of topological observables of strings stretching from u to v. In this case, the boundary deformation parameters s^a are replaced by s^i_{uv}, where u, v run over the topological D-branes, while i indexes a basis ψ^i_{uv} of E_{uv} (in fact, $\{s\}$ can be viewed as a parity-changed dual basis to $\{\psi\}$). Treating s^i_{uv} as non-commuting supervariables leads one to replace the free superalgebra $\mathbb{C}\langle\{s^a\}\rangle$ with the associative superalgebra generated[2] by s^i_{uv} with the relations:

$$s^i_{uv} s^j_{v'w} = 0 \qquad \text{unless } v = v' \,. \tag{1.3}$$

This is the the path algebra $A_{\mathcal{Q}}$ of a quiver $\mathcal{Q}$ whose vertices are the D-branes u, and whose arrows from u to v are the index triples (u, v, i) associated to s^i_{uv}. This quiver is $\mathbb{Z}_2$-graded since s^i_{uv} can be even or odd; as a consequence, $A_{\mathcal{Q}}$ is an associative superalgebra. The case of a single D-brane corresponds to a quiver with a single vertex u, whose path algebra coincides with the free superalgebra generated by the $\mathbb{Z}_2$-graded loops at that vertex.

It is known that path algebras of quivers are formally smooth in the sense of [24], and in fact they provide good non-commutative generalizations of the coordinate rings of smooth affine varieties [15]. One can formulate a non-commutative symplectic geometry for such spaces [25, 26] by extending the construction of [14]. For a general topological D-brane system, the space $\mathbb{A}_{\mathcal{Q}}$ with coordinate ring $A_{\mathcal{Q}}$ is a noncommutative *super*space, whose symplectic form is induced by the parity change of the boundary topological metrics. As for affine superspaces, one finds an even or odd Lie superbracket $\{.,.\}$ which acts on

[2] Over the subalgebra spanned by the trivial paths.

the space $C^0_R(A_{\mathcal{Q}})$ of regular functions on $\mathbb{A}_{\mathcal{Q}}$. The latter can be viewed as the vector superspace spanned by necklaces of the quiver (i.e. cycles of the quiver whose marking by the start=endpoint is forgotten). The generating function W of integrated disk boundary amplitudes is an element of this space, i.e. a linear combination of such necklaces; its parity is opposite that of the boundary topological metrics. The categorical A_∞ constraints for the entire collection of boundary operators amount to equation (1.1). The existence of unit observables in the boundary-preserving sectors can be expressed as a differential constraint on W. Once again, a noncommutative moduli space can be constructed as the invariant theory quotient of the noncommutative critical variety $\mathcal{Z}$ through the symmetries of the system. The existence of boundary unit observables implies that this quotient can be be viewed as a noncommutative 'symplectic reduction', followed by imposing further constraints.

The paper is organized as follows. In section 2, we describe the algebraic structure of finite D-brane systems, starting from the formulation in terms of A_∞ categories found in [7, 8] (which is summarized in appendix A). Upon introducing a finite-dimensional commutative semisimple algebra R, we show that one can express the boundary sector decomposition of [23] as an R-superbimodule structure on the total space $E = \oplus_{u,v}\mathrm{Hom}(u, v)$ of boundary observables. As a consequence, the cyclic and unital weak A_∞ category determined by integrated string correlators can be described as a cyclic and unital weak A_∞ algebra on the superbimodule E. We also summarize the strategy which will be followed in later sections in order to encode this data into a noncommutative generating function. The main step is passing from the superbimodule E to the tensor algebra $A = T_R(E[1]^{\mathrm{v}})$ of its parity-changed dual, which is a superalgebra over R. The boundary topological metrics of the theory induce an (even or odd) noncommutative symplectic form on A. This leads us to study the noncommutative symplectic geometry of R-superalgebras, a subject which is addressed in section 3. In section 4, we apply this to the tensor algebra A, discussing the realization of the abstract objects introduced previously. Upon picking appropriate bases in the space of boundary observables, we show that A can be identified with the path algebra of a $\mathbb{Z}_2$-graded quiver, and give coefficient expressions for the various quantities of interest. We also discuss the quiver version of cyclic derivatives and the so-called loop partial derivatives, two types of operators acting on the space of noncommutative functions. In section 5, we apply the machinery developed in sections 3 and 4 to finite D-brane systems. Using the results of section 2, we show that the cyclic and unital weak A_∞ structure on E is encoded faithfully by a noncommutative generating function W, which can be expressed as a linear combination of quiver necklaces. After discussing the constraints obeyed by W, we say a few words about deformations of the underlying string theory (as opposed to deformations of the boundary data). In section 6, we discuss symmetries of the topological D-brane system and give the algebraic construction of the noncommutative moduli space. Section 7 gives the realization of our construction in the case of a single D-brane. In section 8, we discuss the simplest examples with an even and odd boundary topological metric, and a rather general class of examples with odd boundary metrics, some particular cases of which appeared recently in [27]. Section 9 contains our conclusions.

JHEP11(2005)032

Conventions. Unless specified otherwise, an algebra means an associative and unital algebra over the complex numbers. All morphisms of algebras are assumed to be unital. We will often encounter $\mathbb{Z}\times\mathbb{Z}_2$-graded algebras. To fix the sign convention for such algebras, one must chose a $\mathbb{Z}_2$-valued pairing on the abelian group $\Gamma := \mathbb{Z} \times \mathbb{Z}_2$, i.e. a biadditive and symmetric map $\cdot : \Gamma \times \Gamma \to \mathbb{Z}_2$ specifying how the Koszul rule is applied to bigraded objects. Namely, one agrees that commuting two objects of bidegrees (n, α) and (m, β) always produces the sign $(n, \alpha) \cdot (m, \beta)$. In the present paper, we work with the following choice of pairing:

$$(n, \alpha) \cdot (m, \beta) = [mn] + \alpha\beta \,, \tag{1.4}$$

where here and in the rest of the paper the notation $[k]$ for $k \in \mathbb{Z}$ stands for the mod 2 reduction of k. If α is an element of $\mathbb{Z}_2$, we let $(-1)^\alpha = +1$ if $\alpha = 0$ and -1 if $\alpha = 1$. We let $[1]$ be the parity change functor on the category of super-vector spaces and Σ_U the suspension operator of the supervector space U (i.e. $\Sigma : U \to U[1]$ is the identity operator of U, viewed as an odd map from U to itself). We have $\Sigma_U^2 = \mathrm{id}_U$ and $[1]^2 \approx \mathrm{Id}$, where Id in the last relation is the identity functor. We will often write Σ without indicating the vector space on which it acts, since the latter is usually clear from the context. Given a ring R, we let $\mathrm{Mod-R}$, $\mathrm{R-Mod}$ and $\mathrm{R-Mod-R}$ and $\mathrm{Smod-R}$, $\mathrm{R-Smod}$ and $\mathrm{R-Smod-R}$ be the categories of left-, right- and bi- modules, respectively supermodules over R. For a pair U, V of objects of any of these categories, we let $\mathrm{Hom}(U, V)$ be the space of morphisms from U to V in that category. For supermodules, this consists of degree zero R-linear maps and is an ordinary (i.e. ungraded) module. For two supermodules, we also let $\underline{\mathrm{Hom}}(U, V)$ be the space of morphisms in the corresponding category $\mathrm{Mod-R}$, $\mathrm{R-Mod}$ or $\mathrm{R-Mod-R}$ of *ungraded* objects, obtained by forgetting the $\mathbb{Z}_2$-grading of U, V. The latter consists of all linear maps, without degree conditions; it is known as the *inner* morphism space and is $\mathbb{Z}_2$-graded. In fact, we have $\underline{\mathrm{Hom}}(U, V) = \underline{\mathrm{Hom}}^0(U, V) \oplus \underline{\mathrm{Hom}}^1(U, V)$, where the degree zero component coincides with the space of degree zero maps, $\underline{\mathrm{Hom}}^0(U, V) = \mathrm{Hom}(U, V)$. We will use the same notational convention for the endomorphism and automorphism spaces, for example End(U) is the endomorphism space of U as a supermodule and $\underline{\mathrm{End}}(U)$ its inner endomorphism space etc. Given an R-superbimodule U, we set $T_R U := \oplus_{n\geq 0} U^{\otimes_R n}$ (with $U^{\otimes 0} := R$), the tensor algebra of U, viewed as an $\mathbb{N} \times \mathbb{Z}_2$ graded algebra.

Note. Throughout this paper, an A_∞ algebra or category will mean an A_∞ algebra/category whose sequence of defining products (r_n) terminates (i.e. $r_n = 0$ for sufficiently large n). This condition is purely technical, being needed if one wishes to pass to the dual of a coalgebra in naive manner. The condition can be removed in standard fashion, by considering profinite modules and formal algebras and replacing the relevant maps by continuous maps; then the non-commutative geometric objects mentioned above can be understood in the formal sense. Because most of our considerations generalize straightforwardly to formal case, we adopted the convention of sometimes writing finite sums without indicating the upper bound; in this paper, it is understood that all such sums terminate. When we write a finite sum this manner, it is implied that the sum can be extended to a series in the formal theory.

JHEP11(2005)032

2. Algebraic description of finite topological D-brane systems

In this section, we consider finite D-brane systems in an open topological string theory, i.e. finite collections of D-branes, together with the spaces of boundary observables of all topological strings stretched between them. We will assume that the total space of zero-form boundary observables is finite-dimensional, which is the usual case for topological strings. Using a description derived in [8, 7], we will encode the information of all tree-level boundary string amplitudes into an algebraic structure on a certain superbimodule built out of the total space of zero-form boundary observables.

The results of [8, 7] imply that D-brane systems in topological string theory are described by A_∞ categories. The precise statement is as follows (see appendix A for mathematical background):

A topological D-brane system is described by a weak, cyclic and unital A_∞ category $\mathcal{A}$.

In general, the A_∞ category is only $\mathbb{Z}_2$-graded. This grading can be lifted to a $\mathbb{Z}$-grading provided that the relevant U(1) symmetry of the worldsheet theory is non-anomalous. The objects of $\mathcal{A}$ are the D-branes themselves, while the morphism space $\mathrm{Hom}_{\mathcal{A}}(u,v)$ for two objects u,v is the complex supervector space of boundary zero-form observables for the topological string stretching from u to v. The degree of such observables is given by worldsheet Grassmann parity, and will be denoted by $|.|$. With respect to this grading, the A_∞ products have degrees $[n]$, where the square bracket indicates mod 2 reduction. As in appendix A, it is convenient to work with the parity-changed spaces $\mathrm{Hom}_{\mathcal{A}}(u,v)[1]$, the degree of whose elements we denote by a tilde. Hence $\tilde{x} = [|x|+1]$ for all homogeneous morphisms x. The unitality condition involves even elements 1_u of the endomorphism spaces $\mathrm{Hom}_{\mathcal{A}}(u,u)$ (equivalently, their odd suspensions $\lambda_u := \Sigma 1_u \in \mathrm{Hom}_{\mathcal{A}}(u,u)[1]$), while the cyclicity constraint involves non-degenerate $\mathbb{Z}_2$-homogeneous pairings (the boundary topological metrics) $\rho_{uv} : \mathrm{Hom}_{\mathcal{A}}(u,v) \times \mathrm{Hom}_{\mathcal{A}}(v,u) \to \mathbb{C}$ of common degree $\tilde{\omega} \in \mathbb{Z}_2$, obeying the graded-symmetry condition $\rho_{uv}(x,y) = (-1)^{|x||y|}\rho_{vu}(y,x)$ for homogeneous elements $x \in \mathrm{Hom}_{\mathcal{A}}(u,v)$ and $y \in \mathrm{Hom}_{\mathcal{A}}(v,u)$. The latter can also be expressed in terms of the graded-antisymmetric forms $\omega_{uv} := \rho_{uv} \circ \Sigma^{\otimes 2} : \mathrm{Hom}_{\mathcal{A}}(u,v)[1] \times \mathrm{Hom}_{\mathcal{A}}(v,u)[1] \to \mathbb{C}$ obtained from the topological metrics via suspension. The detailed formulation of this structure (which arises by introducing boundary sectors in the derivation of [7]) is given in appendix A.

Observation. It was argued in [8] (see [28, 29] for the dG case) that the A_∞ category $\mathcal{A}_{full}$ obtained by considering 'all' D-branes of a topological string theory must be endowed with a parity-change functor and be equivalent with its category of twisted complexes. This encodes the physical requirement that the collection of all topological D-branes of a given theory is closed under formation of topological D-brane composites. This 'quasiunitarity constraint' implies that the cohomology category $H^0(\mathcal{A}_{full})$ is an enhanced triangulated category (with an A_∞ enhancement). In the present paper, we work with a fixed A_∞ sub-category $\mathcal{A}$ of the full D-brane category $\mathcal{A}_{full}$, so $\mathcal{A}$ need not obey this supplementary condition (which is impossible to satisfy with a finite collection of objects).

JHEP11(2005)032

For the remainder of this paper, we focus on finite D-brane systems, which means that $Ob\mathcal{A}$ is a finite set and all morphism spaces $\mathrm{Hom}(u, v)$ are finite-dimensional. In this case, we say that the A_∞ category $\mathcal{A}$ is *finite.*

2.1 Encoding the boundary sector decomposition

One can encode the categorical data of $\mathcal{A}$ in an equivalent, but more amenable form. Setting $\mathcal{Q}_0 := Ob\mathcal{A}$, consider the $\mathbb{Z}_2$-graded vector space $E := \oplus_{u,v\in\mathcal{Q}_0}\mathrm{Hom}(u, v)$, which is known as the total boundary space of the D-brane system [23].

Definition 1. *A* binary homogeneous decomposition *over the set* $\mathcal{Q}_0$ *is a pair* $(U, (U_{uv})_{u,v\in\mathcal{Q}_0})$ *such that* U, U_{uv} *are finite-dimensional complex supervector spaces and* $U = \oplus_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0}U_{uv}$ *is a homogeneous decomposition of* U *indexed by the cartesian product* $\mathcal{Q}_0 \times \mathcal{Q}_0$*. The* opposite *of the binary decomposition* $(U, (U_{uv})_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0})$ *is the binary homogeneous decomposition* $(U, (U^{\mathrm{opp}}_{uv})_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0})$*, where* $U^{\mathrm{opp}}_{uv} = U_{vu}$*.*

A morphism of binary homogeneous decompositions over $\mathcal{Q}_0$ from $(U, (U_{uv})_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0})$ to $(U', (U'_{uv})_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0})$ is a degree zero linear map $\phi : U \to U'$ such that $\phi(U_{uv}) \subset U'_{uv}$ for all $(u, v) \in \mathcal{Q}_0 \times \mathcal{Q}_0$. With this definition, binary homogeneous decompositions over $\mathcal{Q}_0$ form a category. A morphism ϕ in this category is an isomorphism iff it is a bijective map. In this case, we have $\phi(U_{uv}) = U'_{uv}$ for all $u, v \in \mathcal{Q}_0$.

A topological D-brane system determines two opposite decompositions of its total boundary space, namely $E = \oplus_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0}\mathrm{Hom}_{\mathcal{A}}(u, v)$ and $E = \oplus_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0}\mathrm{Hom}_{\mathcal{A}}(v, u)$. We will view the first of these as fundamental, so we set $E_{uv} := \mathrm{Hom}_{\mathcal{A}}(u, v)$; then $E^{\mathrm{opp}}_{uv} := \mathrm{Hom}_{\mathcal{A}}(v, u)$. This is consistent with the convention that morphisms compose forward in the definition of an A_∞ category given in appendix A. The binary homogeneous decomposition $(E, (E_{uv})_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0})$ is the so-called *boundary sector decomposition* of the topological D-brane system [23].

Let us consider the finite-dimensional semisimple commutative algebra $R = \oplus_{u\in\mathcal{Q}_0}\mathbb{C}$, where the right hand side is a direct sum of copies of $\mathbb{C}$, viewed as an algebra over itself. We let ϵ_u $(u \in \mathcal{Q}_0)$ be the commuting idempotents of R corresponding to the canonical basis elements of the vector space $\mathbb{C}^{\mathcal{Q}_0}$, so $R = \oplus_{u\in\mathcal{Q}_0}\mathbb{C}\epsilon_u$ with $\epsilon_u\epsilon_v = \delta_{uv}\epsilon_u$. The unit of R is $1_R = \sum_{u\in\mathcal{Q}_0}\epsilon_u$. It is not hard to see that a binary homogeneous decomposition $U = \oplus_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0}U_{uv}$ amounts to giving an R-superbimodule structure on U. Indeed, such a decomposition amounts to giving degree zero commuting idempotents $\epsilon^l_u, \epsilon^r_u \in \mathrm{End}_{\mathbb{C}}(\mathrm{U})$ for all $u \in \mathcal{Q}_0$, namely the projectors on the subspaces $\oplus_v U_{uv}$ and $\oplus_v U_{vu}$ respectively (then $U_{uv} = \epsilon^l_u\epsilon^r_v(U) = \epsilon^r_v\epsilon^l_u(U)$). Notice that U_{uv} can be recovered as $U_{uv} = \epsilon_u U \epsilon_v$ from knowledge of the R-superbimodule structure. Moreover, a morphism of binary homogeneous decompositions over $\mathcal{Q}_0$ amounts to a morphism of R-superbimodules. Hence the category of binary homogeneous decompositions over $\mathcal{Q}_0$ is equivalent with the category of superbimodules over R.

Since R is commutative, any R-superbimodule U defines another R-superbimodule U^{opp} whose underlying supervector space coincides with that of U but whose external multiplications are given by:

$$\alpha * x * \beta = \beta x \alpha \qquad \forall \alpha, \beta \in R \quad \forall x \in U\,.$$

The relations $\epsilon_u * E * \epsilon_v = \epsilon_v E \epsilon_u$ show that the binary decomposition determined by $U^{\rm opp}$ is the opposite of the binary decomposition determined by U:

$$(U^{\rm opp})_{uv} = U^{\rm opp}_{uv} = U_{vu}\,.$$

Applying this to a topological D-brane system, we find that the boundary sector decomposition $E = \oplus_{(u,v)\in \mathcal{Q}_0\times\mathcal{Q}_0} E_{uv} = \oplus_{(u,v)\in \mathcal{Q}_0\times\mathcal{Q}_0} {\rm Hom}_{\mathcal{A}}(u,v)$ is encoded by an R-superbimodule structure on E. Moreover, the opposite decomposition $E = \oplus_{(u,v)\in \mathcal{Q}_0\times\mathcal{Q}_0}$ ${\rm Hom}_{\mathcal{A}}(v,u)$ is encoded by the opposite superbimodule $E^{\rm opp}$. These observations allow one to encode the category-theoretic structure of A_∞ products into compatibility with the R-superbimodule structure of E. Before stating the relevant result, we need a few more preparations.

Given an R-superbimodule U, its dual $U^{\rm v} := {\rm Hom}_{\rm R-mod}(U,R) = \underline{\rm Hom}_{\rm R-Smod}(U,R)$ as a left R-module becomes an R-superbimodule with respect to the external multiplications defined through:

$$(\alpha f \beta)(x) := f(\alpha x \beta) = \alpha f(x\beta)\,. \tag{2.1}$$

We warn the reader that the usual definition of an R-superbimodule structure on $U^{\rm v}$ corresponds to the opposite of that given in (2.1). We adopted the convention (2.1) in order to avoid notational morass later on. With this definition, some of the usual isomorphisms involve taking the opposite of certain superbimodules, as we explain in appendix B; in return, the formulas of sections 4, 5 and 6 simplify considerably.

It is not hard to see that the binary homogeneous decomposition $U^{\rm v} = \oplus_{(u,v)\in \mathcal{Q}_0\times\mathcal{Q}_0}$ $(U^{\rm v})_{uv}$ determined by the R-superbimodule structure (2.1) has components:

$$(U^{\rm v})_{uv} = (U_{uv})^*\,, \tag{2.2}$$

where $(U_{uv})^* := \underline{\rm Hom}_{\mathbb{C}}(U_{uv},\mathbb{C})$ is the linear dual of U_{uv} viewed as a supervector space. Notice that there is no reversal of the positions of u and v in relation (2.2).

With the definition (2.1), a homogeneous R-bilinear form[3] $\sigma : U\times U \to R$ of degree $\tilde\sigma$ induces an R-superbimodule map $U^{\rm opp} \stackrel{j_\sigma}{\to} U[\tilde\sigma]^{\rm v}$ given by $x \to f_x(.) := \sigma(\cdot, x)$. The form is called *graded-symmetric* if $\sigma(x,y) = (-1)^{{\rm deg}x{\rm deg}y}\sigma(y,x)$ for all homogeneous $x,y\in U$ and *graded-antisymmetric* if $\sigma(x,y) = (-1)^{1+{\rm deg}x{\rm deg}y}\sigma(y,x)$. In any of these case, it is called *nondegenerate* if the map $x \to f_x$ is an isomorphism of vector spaces.

A graded-symmetric form ρ on U induces a graded-antisymmetric form $\omega = \rho\circ\Sigma^{\otimes 2}$ on $U[1]$, given explicitly by:

$$\omega(\Sigma x, \Sigma y) = (-1)^{\tilde x}\rho(x,y) \qquad \forall x,y\in U\,,$$

where the sign prefactor is due to the Koszul rule (the suspension operator Σ is odd). If ρ is homogeneous, then ω is homogeneous of the same $\mathbb{Z}_2$ degree. Hence giving a graded-symmetric form on U is equivalent to giving a graded-antisymmetric form on $U[1]$. Moreover, ρ is non-degenerate iff ω is. A non-degenerate, homogeneous and graded-symmetric

[3]Recall that a multilinear map f of R-superbimodules is required to satisfy $f(\alpha x_1, x_2\ldots x_{n-1}, x_n\beta) = \alpha f(x_1\ldots x_n)\beta$ as well as the balance condition $f(x_1\ldots x_{j-1}\alpha, x_j\ldots x_n) = f(x_1\ldots x_{j-1}, \alpha x_j \ldots x_n)$ for all j, where $\alpha,\beta\in R$. In particular, a bilinear map $\sigma : U\times U \to R$ satisfies $\sigma(x\alpha, y) = \sigma(x,\alpha y)$ and $\sigma(\alpha x, y\beta) = \alpha\sigma(x,y)\beta$.

R-bilinear form will be called a *metric*, while a non-degenerate, homogeneous and graded-antisymmetric R-bilinear form will be called a *symplectic form.* A R-superbimodule is called a *metric superbimodule* if it is endowed with a metric, and a *symplectic superbimodule* if it is endowed with a symplectic form. Metric and symplectic R-superbimodules form categories if one defines morphisms in the obvious fashion. Notice that parity change induces an idempotent equivalence between these categories. This reflects the general principle that metric and symplectic superdata are related through parity change. In particular, $(U[1], \omega)$ is a symplectic R-superbimodule iff (U, ρ) (with $\omega = \rho \circ \Sigma^{\otimes 2}$) is a metric R-superbimodule.

Giving an R-bilinear form σ on U amounts to giving $\mathbb{C}$-bilinear forms $\sigma_{uv} : U_{uv} \times U_{vu} \to \mathbb{C}$ for all u, v. Indeed, R-bilinearity of σ implies:

$$\sigma(x\epsilon_v, y) = \sigma(x, \epsilon_v y) \qquad \text{and} \qquad \sigma(\epsilon_u x, y\epsilon_w) = \sigma(x, y)\epsilon_u\epsilon_w \,.$$

Writing $x = \sum_{u,v} x_{uv}$ and $y = \sum_{u,v} y_{uv}$ with $x_{uv} := \epsilon_u x \epsilon_v \in U_{uv}$ and $y_{uv} := \epsilon_u y \epsilon_v \in U_{uv}$, this shows that $\sigma(x_{uv}, y_{u'v'})$ vanishes unless $v = u'$ and $u = v'$. Hence σ is completely determined by its restrictions σ'_{uv} to the subspaces $U_{uv} \times U_{vu}$. Explicitly, we have $\sigma(x, y) = \sum_{u,v} \sigma'_{uv}(x_{uv}, y_{vu})$. Each restriction takes $U_{uv} \times U_{vu}$ into the one-dimensional subspace $\mathbb{C}\epsilon_u$ of R, so we can write $\sigma'_{uv} = \sigma_{uv}\epsilon_u$ for some complex-linear maps $\sigma_{uv} : U_{uv} \times U_{vu} \to \mathbb{C}$. Then:

$$\sigma(x, y) = \sum_{u,v} \sigma_{uv}(x_{uv}, y_{vu})\epsilon_u \tag{2.3}$$

and σ is completely determined by σ_{uv}. Conversely, any family of complex-bilinear maps $\sigma_{uv} : U_{uv} \times U_{vu} \to \mathbb{C}$ determines an R-bilinear map σ through relation (2.3). The map $U \xrightarrow{j_\sigma} U^{\mathrm{v}}$ induced by σ has the property $j_\sigma(U_{uv}^{\mathrm{opp}}) \subset (U^{\mathrm{v}})_{uv}$ i.e. $j_\sigma(U_{vu}) \subset (U_{uv})^*$, and its restrictions to the subspaces U_{vu} can be identified with the maps $U_{vu} \to (U_{uv})^*$ induced by σ_{uv}. Thus σ is non-degenerate iff σ_{uv} are (the later means that all maps $U_{vu} \to (U_{uv})^*$ determined by σ_{uv} are linear isomorphisms). It is also clear that σ is graded-symmetric iff $\sigma_{uv}(x, y) = (-1)^{\tilde{x}\tilde{y}}\sigma_{uv}(y, x)$ for all homogeneous $x \in U_{uv}$ and $y \in U_{vu}$; a similar statement holds for graded-antisymmetric forms.

For a topological D-brane system, we have $E_{uv} = \mathrm{Hom}_{\mathcal{A}}(u, v)$ and the topological metrics $\rho_{uv} : \mathrm{Hom}_{\mathcal{A}}(u, v) \times \mathrm{Hom}_{\mathcal{A}}(v, u) \to \mathbb{C}$ are homogeneous of common degree $\tilde{\omega}$, so they determine an R-bilinear form ρ on E of the same degree. Moreover, the graded-symmetry and non-degeneracy conditions for ρ_{uv} amount to the condition that ρ is a superbimodule metric on E. Similarly, the symplectic forms $\omega_{uv} = \rho_{uv} \circ \Sigma^{\otimes 2}$ determine a superbimodule symplectic form on $E[1]$ of degree $\tilde{\omega}$.

A *weak* A_∞ *structure* on a R-superbimodule U is a countable family of odd R-linear maps $r_n : U[1]^{\otimes n} \to U[1]$ (equivalently, odd R-multilinear maps $r_n : U[1]^n \to U[1]$, which we denote by the same letters) with $n \geq 0$, which satisfy the conditions:[4]

$$\sum_{0 \leq i+j \leq n} (-1)^{\tilde{x}_1 + \dots + \tilde{x}_i} r_{n-j+1}(x_1 \cdots x_i, r_j(x_{i+1} \dots x_{i+j}), x_{i+j+1} \dots x_n) = 0 \tag{2.4}$$

[4]There is some ambiguity in the sign conventions for various objects related to A_∞ algebras and categories. In this paper, we view the homological derivation Q discussed in section 5 as the fundamental object, and have defined r_n such that most signs simplify. We refer the reader to [31] for a discussion of other conventions.

for all $n \geq 0$. In these relations, it is understood that $x_1, \ldots, x_n$ are arbitrary homogeneous elements of $U[1]$. The structure is called *strong*, if $r_0 = 0$ and *minimal* if $r_0 = r_1 = 0$.

We say that a weak A_∞ structure on U is *unital* if there exists an even element $1 \in U^R$ such that its odd suspension $\lambda = \Sigma 1 \in U[1]^R$ satisfies the following conditions:

$$\begin{aligned}
&r_n(x_1 \ldots x_{j-1}, \lambda, x_{j+1} \ldots x_n) = 0 \qquad \text{for all}\, n \neq 2 \quad \text{and all}\, j \\
&-r_2(\lambda, x) = (-1)^{\tilde{x}} r_2(x, \lambda) = x \,,
\end{aligned} \tag{2.5}$$

for all homogeneous elements x, x_j of U. In this case, 1 is called the *even unit* of the A_∞ structure, while λ will be called the *odd unit.* It is not hard to see that the unit of a unital A_∞ structure is unique. Indeed, given another unit $1'$, set $\lambda' := \Sigma 1'$. Then $r_2(\lambda, \lambda') = -\lambda = -\lambda'$, where we used the second row in (2.5) by viewing either λ or λ' as the unit. This implies $1 = 1'$.

Recall that the *center* U^R of an R-superbimodule U (a.k.a the centralizer of R in U) is the homogeneous sub-bimodule consisting of all central elements $x \in U$, i.e. those elements which satisfy $\alpha x = x\alpha$ for all $\alpha \in R$. In terms of the homogeneous binary decomposition, we have $U^R = \oplus_{u \in \mathcal{Q}_0} U_{uu}$. Thus a central element has the form $x = \oplus_{u \in \mathcal{Q}_0} x_u$, with $x_u = \epsilon_u x \epsilon_u = \epsilon_u x = x \epsilon_u \in U_{uu}$. Since R is semisimple, we have a direct sum decomposition of vector spaces:

$$U = U^R \oplus [R, U] \,, \tag{2.6}$$

where $[R, U]$ is the complex supervector space generated by the commutators $[\alpha, x] = \alpha x - x\alpha$ with $\alpha \in R$ and $x \in U$. This follows as in [24] due to the fact that $\oplus_{u \in Q_0} \epsilon_u \otimes \epsilon_u$ is a so-called separability element. It can also be seen directly by using the identities $x - \sum_u \epsilon_u x \epsilon_u = \frac{1}{2} \sum_u [\epsilon_u, [\epsilon_u, x]]$ and $\epsilon_u [\epsilon_v, x] \epsilon_u = 0$, which hold for any element $x \in U$.

Semisimplicity of R implies that the unit of a unital A_∞ structure on an R-superbimodule U must be central. To see this, let λ be the odd A_∞ unit and consider the central element $\lambda' = \sum_u \epsilon_u \lambda \epsilon_u$. Then the last row in (2.5) implies:

$$r_2(x, \lambda') = \sum_u r_2(x, \epsilon_u \lambda \epsilon_u) = \sum_u r_2(x\epsilon_u, \lambda)\epsilon_u = (-1)^{\tilde{x}} \sum_u x\epsilon_u = (-1)^{\tilde{x}} x$$

and:

$$r_2(\lambda', x) = \sum_u r_2(\epsilon_u \lambda \epsilon_u, x) = \sum_u \epsilon_u r_2(\lambda, \epsilon_u x) = -\sum_u \epsilon_u x = -x \,,$$

where we used R-bilinearity of r_2 and the equation $\sum_u \epsilon_u = 1$. These two equations imply $\lambda = \lambda'$ by the argument used above to show unicity of the A_∞ unit. This shows that λ must be central.

A weak A_∞ structure (r_n) on U is called *cyclic* if U[1] is endowed with a homogeneous symplectic form ω, such that the following relations are satisfied:

$$\omega(x_0, r_n(x_1 \ldots x_n)) = (-1)^{\tilde{x}_0 + \tilde{x}_1 + \tilde{x}_0(\tilde{x}_1 + \cdots + \tilde{x}_n)} \omega(x_1, r_n(x_2 \ldots x_n, x_0)) \,. \tag{2.7}$$

In terms of the metric ρ determined by $\omega = \rho \circ \Sigma^{\otimes 2}$, these relations take the following form, which might be more familiar to some readers:

$$\rho(x_0, r_n(x_1 \ldots x_n)) = (-1)^{\tilde{x}_0(\tilde{x}_1 + \cdots + \tilde{x}_n)} \rho(x_1, r_n(x_2 \ldots x_n, x_0)) \,. \tag{2.8}$$

We can now state a basic equivalence:

Proposition 1. *Giving a finite weak cyclic and unital A_∞ category with object set $\mathcal{Q}_0$ amounts to giving a weak, cyclic and unital A_∞ structure on a $\mathbb{Z}_2$-graded R-superbimodule E of finite complex dimension, over the finite-dimensional semisimple commutative algebra $R = \oplus_{u\in\mathcal{Q}_0}\mathbb{C}\epsilon_u$.*

In view of this proposition, one can *define* finite tree-level topological D-brane systems to be cyclic and unital weak A_∞ structures on some R-superbimodule of finite dimension over $\mathbb{C}$, where R is a finite-dimensional semisimple commutative $\mathbb{C}$-algebra. The decomposition of the system into constituent D-branes and the decomposition of the total boundary space E into boundary sectors E_{uv} are both encoded by the R-superbimodule structure.

Sketch of proof. As explained above, the superbimodule structure on E amounts to a homogeneous decomposition $E = \oplus_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0}E_{uv}$, where $E_{uv} := \mathrm{Hom}_{\mathcal{A}}(u, v)$. The rest of the proof is a lengthy but straightforward check of conditions, showing that compatibility of various maps with the R-superbimodule structure of E allows one to translate superbimodule A_∞ data into the categorical A_∞ data listed in appendix A. We already showed above that giving the categorical bilinear forms ρ_{uv} amounts to giving a metric ρ on this superbimodule. Similarly, giving an R-multilinear map $r_n : E^n \to E$ amounts to giving $\mathbb{C}$-multilinear maps $r_{u_1\dots u_{n+1}} : E_{u_1u_2} \times E_{u_2u_3} \times \cdots \times E_{u_nu_{n+1}} \to E_{u_1u_{n+1}}$, and it is clear that the weak A_∞ constraints for r_n amount to the categorical weak A_∞ constraints for these maps. Moreover, the cyclicity conditions for r_n with respect to $\omega = \rho \circ \Sigma^{\otimes 2}$ amount to the categorical cyclicity constraints with respect to $\omega_{uv} := \rho_{uv} \circ \Sigma^{\otimes 2}$. Finally, the A_∞ units 1_u of $\mathcal{A}$ give an even central element $1 = \oplus_u 1_u$ in the superbimodule E, which is a unit for the superbimodule A_∞ structure. Conversely, giving such a unit amounts to giving elements 1_u in each 'diagonal' subspace E_{uu}, since the unit of an R- superbimodule A_∞ structure must be central. The unitality constraints for r_n amount to the categorical unitality constraints for $r_{u_1\dots u_{n+1}}$.

Observation. When the A_∞ structure (r_n) is minimal, the first non-trivial A_∞ constraint implies that the product $\cdot = \Sigma \circ r_2 \circ \Sigma^{\otimes 2}$ is associative. Moreover, cyclicity and unitality imply that the triple $(E, \cdot, \rho)$ is a (non-commutative) Frobenius algebra, whose multiplication and bilinear form are R-bilinear. On the other hand, forgetting all higher products of $\mathcal{A}$ gives a usual (i.e. associative and unital) category endowed with non-degenerate and graded symmetric bilinear pairings ρ_{uv} between opposite spaces of morphisms. As explained in [23], such a category describes the boundary part of a two-dimensional topological field theory defined on bordered Riemann surfaces. This gives the following:

Corollary 1. *The boundary sector of a topological field theory in two-dimensions, in the presence of a finite system of topological D-branes, is described by a noncommutative Frobenius structure on an R-superbimodule E, i.e. a unital noncommutative Frobenius algebra on the vector superspace E, whose associative product and pairing are R-bilinear.*

2.2 The geometrization strategy

The algebraic formulation of the previous subsection allows one to avoid the notational morass of the category-theoretic description. One is still left with the rather complicated data of a cyclic and unital weak A_∞ structure on E. To express this synthetically, we will use a $\mathbb{Z}_2$-graded version of the non-commutative symplectic geometry of quivers developed in [25, 26]. Much of the content of the next two sections is a relatively straightforward, though tedious, superextension of the construction of [25, 26], so it might be useful to summarize the main points. Start with a cyclic and unital A_∞ structure on the R-superbimodule E. To encode this geometrically, we will proceed as follows:

(1) Giving a weak A_∞ structure on the superbimodule E amounts to giving an odd derivation Q of the tensor algebra $A = T_R E[1]^{\mathrm{v}}$, satisfying the condition $Q^2 = 0$. Hence (A, Q) can be viewed as a noncommutative version of the Q-manifolds considered in [30].

(2) The symplectic form on $E[1]$ induces a non-commutative symplectic form on A, and one can develop the non-commutative symplectic supergeometry of this algebra by extending the approach of [25, 26]. This gives notions of symplectic and hamiltonian superderivations having the classical properties. The Karoubi complex $C_R(A)$ is acyclic in positive degrees so all symplectic superderivations are hamiltonian. There is a $\mathbb{Z}_2$-graded version $\{.,.\}$ of the Kontsevich bracket, a super-Lie bracket on $C^0_R(A)[\tilde{\omega}]$, where $\tilde{\omega}$ is the $\mathbb{Z}_2$-degree of the symplectic form.

(3) Cyclicity of (r_n) amounts to the condition that Q be a symplectic derivation, i.e. $L_Q \omega = 0$, where L_Q is the Lie superderivative along Q. Thus (A, Q, ω) is a noncommutative version of the QP-manifolds of [30].

(4) The non-commutative generating function W of the D-brane system is the hamiltonian of the homological derivation Q; to determine this uniquely, we require that it 'vanishes at zero' in an appropriate sense. Thus W is an element of degree $\tilde{\omega}+1$ of the supervector space $C^0_R(A) = A/[A, A]$. The weak A_∞ constraint $Q^2 = 0 \Leftrightarrow [Q, Q] = 0$ is equivalent with the condition $\{W, W\} = 0$. The superderivation Q can be reconstructed as the hamiltonian derivation determined by W, so the A_∞ structure defined by W is automatically cyclic.

(5) Unitality of the weak A_∞ structure is equivalent to the condition $\frac{1}{2}\overrightarrow{\delta}_\lambda W = \mu$, where μ is a superized version of the moment map of [21] and $\overrightarrow{\delta}_\lambda$ is the cyclic derivative of W with respect to the odd A_∞ unit λ. The noncommutative generating function determines the symplectic form through this relation.

(6) Since $R = \oplus_{u \in \mathcal{Q}_0} \mathbb{C}\epsilon_u$, the tensor algebra A can be viewed as the path algebra of a superquiver $\mathcal{Q}$, a quiver on the vertex set $\mathcal{Q}_0$ endowed with a $\mathbb{Z}_2$-valued map on its set of arrows. The quiver presentation arises by choosing homogeneous bases of the vector space E which are adapted to its binary homogeneous decomposition. Using quiver

language amounts to working in 'special' coordinates on the non-commutative space determined by A, where 'special' means that the coefficients of the non-commutative symplectic form in such coordinates are complex numbers.

(7) All constructions have natural quiver interpretations. For example, $C^0_R(A)$ can be described in terms of necklaces, which in our case are $\mathbb{Z}_2$-graded. Cyclic derivatives with respect to elements of the adapted basis translate into cyclic derivatives with respect to the quiver's arrows a. Relative to an appropriate adapted basis, the unitality constraint amounts to the requirement that W has a certain dependence on an odd element σ of $E[1]^{\mathrm{v}}$ determined by the odd A_∞ unit λ.

(8) One can view A as the coordinate ring of a non-commutative supermanifold $\mathbb{A}_Q$. Imposing the relations $\overrightarrow{\delta}_a W = 0$ gives the non-commutative extended vacuum space $\mathcal{Z}$, a 'noncommutative subscheme' of $\mathbb{A}_Q$. The non-commutative extended moduli space $\mathcal{M}$ is obtained by modding $\mathcal{Z}$ (in the GIT sense) through those symplectomorphisms which correspond to unital and cyclic A_∞ automorphisms of the underlying D-brane category.

We now proceed with the detailed discussion of these points.

3. Noncommutative symplectic geometry of R-superalgebras

In this section, we extend the construction of [25, 26] to the case of superalgebras. The proofs of most statements are straightforward adaptations of those given in [25, 26], so I will only indicate the points where our conventions are important or something interesting happens.

Let R be a unital and commutative algebra over $\mathbb{C}$. An *R-superalgebra* is a unital superalgebra A containing R as a subalgebra in even degrees (notice that this is stronger than requiring that A be a superalgebra over R, since we require that R sits inside the degree zero subalgebra of A). A morphism of R-superalgebras is a morphism of superalgebras whose restriction to R equals the identity map.

3.1 Noncommutative differential superforms

Given an R-superalgebra A, consider the R-superbimodule $A_R := A/R$, where A/R stands for the vector space quotient. We define the *space of relative noncommutative forms of A over R* by $\Omega_R A := \oplus_{n\geq 0}\Omega^n_R A$, where:

$$\Omega^n_R A = A \otimes_R T^n_R(A_R) = A \otimes_R A_R^{\otimes_R n} .$$

We write the elements of this space as $w = a_0 da_1 \dots da_n$, where da is the image of $a \in A$ under the projection $A \xrightarrow{d} A_R = A/R$ and juxtaposition stands for the tensor product over R. The space $\Omega_R A$ is given a differential algebra structure with product:

$$\begin{aligned}
&(a_0 da_1 \dots da_n)(b_0 db_1 \dots db_m) := \\
&\quad := a_0 da_1 \dots da_n d(a_n b_0) db_1 \dots db_m +
\end{aligned}$$

$$+\sum_{i=1}^{n-1}(-1)^i a_0 da_1 \dots d(a_{n-i}a_{n-i+1})\dots da_n db_0 \dots db_m +$$
$$+(-1)^n a_0 a_1 da_2 \dots da_n db_0 \dots db_m$$

and differential $d(a_0da_1 \dots da_n) = da_0da_1 \dots da_n$. In this paper, we view $\Omega_R A$ as an $\mathbb{N}\times\mathbb{Z}_2$ graded algebra, whose $\mathbb{N}$-grading (with components $\Omega_R^n A$) is given by the 'rank' of forms, and whose $\mathbb{Z}_2$ grading is induced from A. We denote the bidegree of homogeneous elements by $\deg w = (\bar{w}, \tilde{w}) \in \mathbb{N}\times\mathbb{Z}_2$. As explained in the introduction, we always work with the pairing (1.4), and will require that d has bidegree $(1,0) \in \mathbb{Z}\times\mathbb{Z}_2$. This means that d has the derivation property:

$$d(w_1w_2) = (dw_1)w_2 + (-1)^{\bar{w}_1} w_1 \cdot dw_2 \qquad \text{for all } w_1, w_2 \in \Omega_R A\,.$$

We stress that in our conventions d has degree *zero* with respect to the $\mathbb{Z}_2$-grading. The detailed construction of $\Omega_R A$ is given in appendix C. The space $\Omega_R^1 A$ has an A-superbimodule structure with multiplications:

$$\alpha(adb)\beta = (\alpha a)d(b\beta) - (\alpha ab)d\beta \qquad \forall a,b\in A, \quad \alpha,\beta\in R\,.$$

As in [24], one has an isomorphism of bigraded algebras $\Omega_R A \approx T_A(\Omega_R^1 A)$, which induces an A-superbimodule structure on $\Omega_R A$. In particular, $\Omega_R A$ is an R-superalgebra (since $R\subset A = \Omega_R^0 A \subset \Omega_R A$).

As usual, the pair $(\Omega_R A, d)$ has a universality property. To formulate it, we define an *R-differential superagebra* to be an $\mathbb{N}\times\mathbb{Z}_2$-graded unital differential algebra (Ω, d) such that $\deg(d) = (1,0)$, $R\subset\Omega^{0,0}$ and $d(R) = 0$, where $\Omega^{0,0}$ is the subspace of elements of vanishing bidegree. If (Ω_j, d_j) are two such algebras, a map $\phi : \Omega_1 \to \Omega_2$ is called a morphism of R-differential superalgebras if:

(1) ϕ is a morphism of unital $\mathbb{N}\times\mathbb{Z}_2$-graded algebras (in particular, ϕ has vanishing bidegree)

(2) $\phi|_R = \mathrm{id}_R$

(3) $d_2\circ\phi = \phi\circ d_1$.

With this definition, R-differential superalgebras form a category. The universality property of $(\Omega_R A, d)$ is as follows. Given any R-differential superalgebra (Ω, d) and a morphism of R-superalgebras $\rho : A\to\Omega$ such that $\rho(A)\subset\Omega^0$, there exists a unique morphism $u : \Omega_R A\to\Omega$ of R-differential superalgebras such that $\rho = uj$, where $j : A\to\Omega_R A$ is the inclusion. Hence $(\Omega_R A, j)$ is an initial object among the pairs (Ω, ρ). In fact, the correspondence $\Omega \to \{\ R\text{-superalgebra morphisms } \rho : A\to\Omega \text{ with } \rho(A)\subset\Omega^0\ \}$ defines a functor from the category of R-differential superalgebras to the category of sets. The universality property means that $(\Omega_R A, d)$ represents this functor, so it is the superdifferential envelope of A. In particular, any morphism $\phi : A_1\to A_2$ of R-superalgebras extends uniquely to a morphism $\phi^* : \Omega_R A_1\to\Omega_R A_2$ of R-differential superalgebras.

JHEP11(2005)032

3.2 Super Lie derivatives and contractions

Recall that a left (right) derivation of A is a derivation of A viewed as a left (right) supermodule over $A\otimes A^{op}$. Thus a homogeneous left derivation D satisfies $D(ab) = (Da)b + (-1)^{\tilde{a}\tilde{D}}aDb$, while a homogeneous right derivation satisfies $(ab)D = a(bD) + (-1)^{\tilde{b}\tilde{D}}(aD)b$, where $\tilde{D}$ is the degree of D and we use the convention that left derivations are written to the left, and right derivations are written to the right. We will sometimes also indicate this by writing arrows above D.

A relative derivation of A is a derivation which is R-linear, i.e. vanishes on the subalgebra R. We let $\mathrm{Der}_l(A)$ and $\mathrm{Der}_r(A)$ be the complex supervector spaces of *relative* left and right derivations of A, viewed as Lie superalgebras with respect to the supercommutator $[D_1, D_2] = D_1 \circ D_2 - (-1)^{\tilde{D}_1\tilde{D}_2}D_2 \circ D_1$, which satisfies $[D_1, D_2] = (-1)^{1+\tilde{D}_1\tilde{D}_2}[D_2, D_1]$. We let $\mathrm{Der}^{\alpha}_{l,r}(A)$ be the subspaces consisting of left and right relative derivations of degree α.

Similarly, let $\mathrm{Der}_{l,r}(\Omega_R A)$ be the $\mathbb{Z} \times \mathbb{Z}_2$-graded complex vector spaces of *relative* left/right derivations of $\Omega_R A$, i.e. those left/right derivations of $\Omega_R A$ which vanish on R. In this definition, we view $\Omega_R A$ as a $\mathbb{Z} \times \mathbb{Z}_2$ graded algebra with the degree pairing (1.4); thus a bihomogeneous left derivation D of $\Omega_R A$ satisfies:

$$D(w_1 w_1) = Dw_1 w_2 + (-1)^{\deg D\cdot \deg w_1} w_1 D w_2\,,$$

while a bihomogeneous right derivation obeys:

$$(w_1 w_2)D = w_1(w_2 D) + (-1)^{\deg D\cdot \deg w_2}(w_1 D)w_2\,.$$

The spaces $\mathrm{Der}_{l,r}(\Omega_R A)$ are bigraded Lie superalgebras with respect to the bigraded supercommutator $[D_1, D_2] = D_1 \circ D_2 - (-1)^{\deg D_1\cdot \deg D_2}D_2 \circ D_1$, which satisfies $[D_1, D_2] = (-1)^{1+\deg D_1\cdot \deg D_2}[D_2, D_1]$. We let $\mathrm{Der}^{\alpha}_{l,r}(\Omega_R A)$ be the subspaces consisting of left relative derivations of bidegree α.

Let $\theta \in \mathrm{Der}_l(A)$ be a homogeneous relative left derivation of degree $\tilde{\theta}$. The *contraction by* θ is the unique relative left derivation $i_\theta \in \mathrm{Der}^{-1,\tilde{\theta}}_l(\Omega_R A)$ which satisfies $i_\theta a = 0$ and $i_\theta(da) = \theta(a)$ for all $a \in A$. The *Lie derivative along* θ is the unique left derivation $L_\theta \in \mathrm{Der}^{0,\tilde{\theta}}_l(\Omega_R A)$ which satisfies $L_\theta a = \theta(a)$ and $L_\theta(da) = d\theta(a)$ for all a. There are obvious versions of these definitions for right derivations. It is easy to check that i_θ and L_θ are well-defined. Let $\mathrm{Aut_R(A)}$ be the space of automorphisms of A as an R-superalgebra (in particular, all maps $\phi \in \mathrm{Aut_R(A)}$ are even and restrict to the identity on the subalgebra R). For any $\theta, \gamma \in \mathrm{Der}_l(A)$ and $\phi \in \mathrm{Aut_R(A)}$, we have the identities:

$$\begin{aligned}
L_\theta &= [i_\theta, d]\\
[L_\theta, i_\gamma] &= i_{[\theta,\gamma]}\\
[L_\theta, L_\gamma] &= L_{[\theta,\gamma]}\\
[i_\theta, i_\gamma] &= 0\\
[L_\theta, d] &= 0\\
L_{\phi\theta\phi^{-1}} &= \phi^* L_\theta \phi^{*-1}\\
i_{\phi\theta\phi^{-1}} &= \phi^* i_\theta \phi^{*-1}\,.
\end{aligned} \tag{3.1}$$

As usual, these follow by noticing that all left and right hand sides are derivations of the same bi-degree on $\Omega_R A$, and checking agreement on the generators a and da ($a \in A$). Given $w = a_0 da_1 \dots da_n$ with $a_i \in A$, we have:

$$\begin{aligned}
i_\theta w &= \sum_{i=1}^{n} (-1)^{i-1+\tilde{\theta}(\tilde{a}_0+\dots+\tilde{a}_{i-1})} a_0 da_1 \dots da_{i-1} \theta(a_i) da_{i+1} \dots da_n \\
L_\theta w &= \theta(a_0) da_1 \dots da_n + \sum_{i=1}^{n} (-1)^{\tilde{\theta}(\tilde{a}_0+\dots+\tilde{a}_{i-1})} a_0 da_1 \dots da_{i-1} d\theta(a_i) da_{i+1} \dots da_n\,.
\end{aligned}$$

3.3 The bigraded Karoubi complex

Consider the $\mathbb{N} \times \mathbb{Z}_2$-graded vector space:

$$C_R(A) := \frac{\Omega_R A}{[\Omega_R A, \Omega_R A]}\,, \tag{3.2}$$

where $[\Omega_R A, \Omega_R A] \subset \Omega_R A$ is the image of the bigraded commutator map $[.,.] : \Omega_R A \times \Omega_R A \to \Omega_R A$:

$$[w_1, w_2] = w_1 w_2 - (-1)^{\deg w_1 \cdot \deg w_2} w_2 w_1\,.$$

Notice that $[\Omega_R A, \Omega_R A]$ is a homogeneous subspace of $\Omega_R A$ but not a subalgebra. We let $\pi : \Omega_R A \to C_R(A)$ be the projection, and use the notation:

$$\pi(w) := (w)_c \qquad \text{for } w \in \Omega_R A\,.$$

We also let $C^n_R(A)$ be the $\mathbb{N}$-homogeneous components of $C_R(A)$.

Any relative derivation of $\Omega_R A$ preserves the subspace $[\Omega_R A, \Omega_R A]$, so it descends to a well-defined linear operator in $C_R(A)$. In particular, d induces a differential $\bar{d}$ on $C_R(A)$. The bigraded vector space $(C_R(A), \bar{d})$ is the *relative Karoubi (or non-commutative de Rham) complex* of A over R. This differential space is $\mathbb{N} \times \mathbb{Z}_2$-graded, and we have $\deg \bar{d} = (1, 0)$. The supervector spaces $H^n_R(A) := H^n_{\bar{d}}(C_R(A))$ are called the *relative de Rham cohomology groups of A*.

Given a morphism of R-superalgebras, the induced map $\phi^* : (\Omega_R A_1, d_1) \to (\Omega_R A_2, d_2)$ of R-differential superalgebras satisfies $\phi^*([\Omega_R A_1, \Omega_R A_1]) \subset [\Omega_R A_2, \Omega_R A_2]$, so it descends to a morphism of bigraded complexes $\bar{\phi}^* : (C_R(A_1), \bar{d}_1) \to (C_R(A_2), \bar{d}_2)$. In particular, any endomorphism ϕ of A induces an endomorphism $\bar{\phi}^*$ of $(C_R(A), \bar{d})$, which is an automorphism if ϕ is. We let $\bar{\phi}$ be the restriction of $\bar{\phi}^*$ to $C^0_R(A) = A/[A, A]$ (here $[A, A] \subset A$ is the image of the supercommutator $[.,.] : A \times A \to A$).

The contraction and Lie operators i_θ, L_θ also descend to well-defined $\mathbb{C}$-linear maps on $C_R(A)$, which we denote by $\bar{i}_\theta$, $\bar{L}_\theta$. It is clear that the induced operators satisfy all properties listed in eqs. (3.1):

$$\begin{aligned}
\bar{L}_\theta &= \left[\bar{i}_\theta, \bar{d}\right] \\
\left[\bar{L}_\theta, \bar{i}_\gamma\right] &= \bar{i}_{[\theta,\gamma]} \\
\left[\bar{L}_\theta, \bar{L}_\gamma\right] &= \bar{L}_{[\theta,\gamma]} \\
\left[\bar{i}_\theta, \bar{i}_\gamma\right] &= 0
\end{aligned}$$

$$
\begin{aligned}
\left[\bar{L}_\theta, \bar{d}\right] &= 0 \\
\bar{L}_{\phi\theta\phi^{-1}} &= \bar{\phi}^* \bar{L}_\theta \bar{\phi}^{*-1} \\
\bar{i}_{\phi\theta\phi^{-1}} &= \bar{\phi}^* \bar{i}_\theta \bar{\phi}^{*-1} \,,
\end{aligned} \tag{3.3}
$$

where $\phi \in \mathrm{Aut_R(A)}$.

3.4 Noncommutative supersymplectic forms

An element $\omega \in C^2_R(A)$ is called non-degenerate if the following complex-linear map is bijective:

$$
\theta \in \mathrm{Der}_l(A) \to \bar{i}_\theta \omega \in C^1_R(A) \,. \tag{3.4}
$$

A *relative non-commutative symplectic form* on A is a $\mathbb{Z}_2$-homogeneous element $\omega \in C^2_R(A)$ which is closed ($\bar{d}\omega = 0$) and non-degenerate.

Given a symplectic form ω of $\mathbb{Z}_2$-degree $\tilde{\omega}$, a relative derivation $\theta \in \mathrm{Der}_l(A)$ is called *symplectic* if $\bar{L}_\theta \omega = 0$. Let $\mathrm{Der}^\omega_l(A) \subset \mathrm{Der}_l(A)$ be the subspace of all symplectic derivations. By the third property in (3.3), this is a (super) Lie subalgebra of $\mathrm{Der}_l(A)$. As in the even case, it is easy to see that the following map is an isomorphism of supervector spaces:

$$
\theta \in \mathrm{Der}^\omega_l(A) \to \bar{i}_\theta \omega \in C^1_R(A)_{\mathrm{closed}}[\tilde{\omega}] \,. \tag{3.5}
$$

Here $C^1_R(A)_{\mathrm{closed}} = \{\eta \in C^1_R(A) | \bar{d}\eta = 0\}$, a homogeneous subspace of $C^1_R(A)$.

This implies that any $f \in C^0_R(A)$ defines a unique element $\theta_f \in \mathrm{Der}^\omega_l(A)$, determined by the equation $\bar{i}_{\theta_f}\omega = \bar{d}f$. Let $\psi_\omega : C^0_R(A) \to \mathrm{Der}^\omega_l(A)$ be the complex-linear map given by $\psi_\omega(f) := \theta_f$. The relation $\bar{i}_\theta \omega = \bar{d}f$ implies $\tilde{\theta}_f = \tilde{\omega} + \tilde{f}$ for any homogeneous f, so the map ψ_ω is homogeneous of degree $\tilde{\omega}$. It is clear that the following sequence of supervector spaces is exact:

$$
0 \to H^0_R(A) \hookrightarrow C^0_R(A) \stackrel{\psi_\omega}{\to} \mathrm{Der}^\omega_l(A)[\tilde{\omega}] \,, \tag{3.6}
$$

where the map in the middle is the inclusion. Elements $\theta \in \mathrm{im}\psi_\omega$ are called *hamiltonian derivations*. Given a hamiltonian derivation θ, an element $f \in C^0_R(A)$ such that $\psi_\omega(f) = \theta \Leftrightarrow \theta = \theta_f$ is called a *hamiltonian* associated with θ. The sequence (3.6) shows that the hamiltonian of a hamiltonian derivation is determined up to addition of elements of $H^0_R(A)$.

3.5 $\mathbb{Z}_2$-graded version of the Kontsevich bracket

We have the following generalization of an operation introduced in [14].

Definition 2. *The* Kontsevich bracket *induced by ω is the $\mathbb{Z}_2$-homogeneous complex-linear map $\{.,.\} : C^0_R(A) \otimes_{\mathbb{C}} C^0_R(A) \to C^0_R(A)$ of degree $\tilde{\omega}$ defined through:*

$$
\{f, g\} := \bar{i}_{\theta_f} \bar{i}_{\theta_g} \omega \qquad \forall f, g \in C^0_R(A) \,, \tag{3.7}
$$

where $\theta_f = \psi_\omega(f)$ and $\theta_g = \psi_\omega(g)$.

Notice the relation:

$$
\{f, g\} = \bar{L}_{\theta_f} g \,. \tag{3.8}
$$

The following result gives the basic properties of the bracket in the $\mathbb{Z}_2$-graded case.

Proposition 2. *The Kontsevich bracket satisfies the identities:*

$$\{g,f\} = (-1)^{1+(\tilde{f}+\tilde{\omega})(\tilde{g}+\tilde{\omega})}\{f,g\}\,. \tag{3.9}$$

and:

$$(-1)^{(\tilde{f}_1+\tilde{\omega})(\tilde{f}_3+\tilde{\omega})}\{f_1,\{f_2,f_3\}\} + (-1)^{(\tilde{f}_2+\tilde{\omega})(\tilde{f}_1+\tilde{\omega})}\{f_2,\{f_3,f_1\}\} + \\ + (-1)^{(\tilde{f}_3+\tilde{\omega})(\tilde{f}_2+\tilde{\omega})}\{f_3,\{f_1,f_2\}\} = 0\,. \tag{3.10}$$

Hence $(C^0_R(A)[\tilde{\omega}], \{.,.\})$ is a Lie superalgebra.

Proof. The first property follows immediately from $[\bar{i}_{\theta_f}, \bar{i}_{\theta_g}] = 0$. For the second property, let $f_1, f_2, f_3 \in C^0_R(A)$ and $\theta_i := \phi_\omega(f_i)$. Identities (3.3) give:

$$\begin{aligned} 0 = \bar{i}_{\theta_2}\bar{i}_{\theta_1}\bar{i}_{\theta_0} d\omega \,=\, & (-1)^{\tilde{\theta}_0(\tilde{\theta}_1+\tilde{\theta}_2)}\bar{L}_{\theta_0}\bar{i}_{\theta_2}\bar{i}_{\theta_1}\omega + (-1)^{1+\tilde{\theta}_1\tilde{\theta}_2}\bar{L}_{\theta_1}\bar{i}_{\theta_2}\bar{i}_{\theta_0}\omega + \bar{L}_{\theta_2}\bar{i}_{\theta_1}\bar{i}_{\theta_0}\omega + \\ & + (-1)^{1+\tilde{\theta}_0\tilde{\theta}_1}\bar{i}_{\theta_2}\bar{i}_{[\theta_0,\theta_1]}\omega + (-1)^{\tilde{\theta}_2(\tilde{\theta}_0+\tilde{\theta}_1)}\bar{i}_{\theta_1}\bar{i}_{[\theta_0,\theta_2]}\omega + \\ & + (-1)^{1+\tilde{\theta}_0\tilde{\theta}_1+\tilde{\theta}_2(\tilde{\theta}_0+\tilde{\theta}_1)}\bar{i}_{\theta_0}\bar{i}_{[\theta_1,\theta_2]}\omega\,. \end{aligned}$$

Using the relations $[\theta_i,\theta_j] = (-1)^{1+\tilde{\theta}_i\tilde{\theta}_j}[\theta_j,\theta_i]$ and the properties of $\bar{L}_{\theta_i}$, the right hand side can be brought to the form:

$$2\left[(-1)^{1+\tilde{\theta}_0\tilde{\theta}_1+\tilde{\theta}_1\tilde{\theta}_2+\tilde{\theta}_2\tilde{\theta}_0}\{f_0,\{f_1,f_2\}\}+(-1)^{1+\tilde{\theta}_1\tilde{\theta}_2}\{f_1,\{f_2,f_0\}\}+(-1)^{1+\tilde{\theta}_0\tilde{\theta}_1}\{f_2,\{f_0,f_1\}\}\right]. \tag{3.11}$$

Multiplying with $(-1)^{1+\tilde{\theta}_0\tilde{\theta}_1+\tilde{\theta}_1\tilde{\theta}_2}$ leads to the identity:

$$(-1)^{\tilde{\theta}_0\tilde{\theta}_2}\{f_0,\{f_1,f_2\}\} + (-1)^{\tilde{\theta}_1\tilde{\theta}_0}\{f_1,\{f_2,f_0\}\} + (-1)^{\tilde{\theta}_2\tilde{\theta}_1}\{f_2,\{f_0,f_1\}\} = 0\,.$$

This implies equation (3.10) upon changing the indices $0,1,2$ into $1,2,3$ and using $\tilde{\theta}_i = \tilde{f}_i + \tilde{\omega}$. The map ψ_ω has another property which parallels classical behavior.

Proposition 3. *We have:*

$$\theta_{\{f,g\}} = [\theta_f, \theta_g] \qquad \forall f,g \in C^0_R(A)\,. \tag{3.12}$$

Thus $\psi_\omega : (C^0_R(A)[\tilde{\omega}], \{.,.\}) \to (\mathrm{Der}^\omega_l(A), [.,.])$ is a morphism of Lie superalgebras over $\mathbb{C}$.

Proof. Compute:

$$\bar{d}\{f,g\} = \bar{d}\bar{i}_{\theta_f}\bar{i}_{\theta_g}\omega = \bar{L}_{\theta_f}\bar{i}_{\theta_g}\omega + \bar{i}_{\theta_f}\bar{d}(\bar{i}_{\theta_g}\omega) = \bar{L}_{\theta_f}\bar{i}_{\theta_g}\omega = [\bar{L}_{\theta_f}, \bar{i}_{\theta_g}]\omega = \bar{i}_{[\theta_f,\theta_g]}\omega\,.$$

In the third equality, we used $\bar{i}_{\theta_g}\omega = \bar{d}g$ and $\bar{d}^2 = 0$, while in the fourth we used $\bar{L}_{\theta_f}\omega = 0$.

An R-superalgebra automorphism $\phi \in \mathrm{Aut_R(A)}$ is called a *relative symplectomorphism* if $\bar{\phi}^*(\omega) = \omega$. We let $\mathrm{Aut^\omega_R(A)} \subset \mathrm{Aut_R(A)}$ be the subgroup of relative symplectomorphisms of A. By the sixth property in (3.3), the obvious action of $\mathrm{Aut^\omega_{alg}(A)}$ on $\mathrm{Der}_l(A)$ preserves the Lie subalgebra $\mathrm{Der}^\omega_l(A)$. Given a symplectomorphism ϕ, the last property in (3.3) implies $\phi \circ \theta_f \circ \phi^{-1} = \theta_{\bar{\phi}(f)}$ for $f \in C^0_R(A)$, i.e. $\psi_\omega \circ \bar{\phi} = \mathrm{Ad}_\phi \circ \psi_\omega$ for all $\phi \in \mathrm{Aut^\omega_R(A)}$. In turn, this gives $\{\bar{\phi}(f), \bar{\phi}(g)\} = \bar{\phi}(\{f,g\})$. Hence $\mathrm{Aut^\omega_R(A)}$ acts on $(A[\tilde{\omega}], \{.,.\})$ by Lie algebra automorphisms.

4. Noncommutative calculus for finite D-brane systems

Consider a finite D-brane system with object set $\mathcal{Q}_0$. As in section 2, we set $R = \oplus_{u\in\mathcal{Q}_0}\mathbb{C}\epsilon_u$ and let E be the R-superbimodule of boundary sectors. Recall that $E[1]$ carries a symplectic form $\omega = \rho\circ\Sigma^{\otimes 2}$, the parity change of the topological metric ρ. Setting $V = E[1]^{\mathrm{v}}$, we consider the $\mathbb{N}\times\mathbb{Z}_2$ graded tensor algebra $A = T_R(V)$ whose $\mathbb{N}$-homogeneous subspaces we denote by $A_n = T_R^n(V)$. With respect to its $\mathbb{Z}_2$-grading, A is an R-superalgebra with $R = A_0$.

4.1 Generalities

The symplectic form ω on $E[1]$ induces a relative noncommutative symplectic form ω_{form} on A as follows. Recall from section 2 that the symplectic form ω defines a map $j_\omega : E[1]^{\mathrm{opp}} \to E[1]^{\mathrm{v}}$ which is a morphism of R-bimodules, i.e. an element of $\underline{\mathrm{Hom}}(E[1]^{\mathrm{opp}}, E[1]^{\mathrm{v}})$. As explained in appendix B, there exists an isomorphism of R-superbimodules between $\underline{\mathrm{Hom}}$ $(E[1]^{\mathrm{opp}}, E[1]^{\mathrm{v}})$ and the center of the superbimodule $E[1]^{\mathrm{v}}\otimes_R E[1]^{\mathrm{v}}$, which allows us to view ω as an element $\hat{\omega}$ of degree $\tilde{\omega}$ of the space $(V\otimes_R V)^R$. If $\hat{\omega} = \sum_i f_i\otimes_R g_i$ with $f_i, g_i \in V$, then it is shown in appendix B that ω can be recovered as:

$$\omega(x,y) = \sum_i f_i(xg_i(y))\,. \tag{4.1}$$

Using the element $\hat{\omega}$, we define a noncommutative two-form on $A = T_R V$ through the relation:

$$\omega_{\mathrm{form}} = -\frac{1}{2}\sum_i (df_i dg_i)_c \in C_R^2(A)\,, \tag{4.2}$$

where the minus sign is introduced for later convenience. It is easy to see that ω_{form} is well-defined and of the same $\mathbb{Z}_2$-degree as ω. Moreover, it is not hard to check that this two-form is symplectic.

Since $A = T_R V$, the algebra of noncommutative forms $\Omega_R A$ has a second $\mathbb{N}$-grading, which is induced from the $\mathbb{N}$-grading of A (with respect to this grading, we have $\deg a = \deg da = 1$ for all $a\in V$, while R sits in degree zero). This induces a similar $\mathbb{N}$-grading on $C_R(A)$. We let $(\Omega_R A)_n$ and $C_R(A)_n$ be the homogeneous subspaces determined by this grading. A *constant* noncommutative two-form [5] on A is an element of $C_R^2(A)_2$. Thus (4.2) is a constant symplectic form, and any constant symplectic form on A has such an expansion. We let $CNS^{\tilde{\omega}}(V)$ be the vector space of all constant noncommutative symplectic forms on $T_R V$ of degree $\tilde{\omega}$. In what follows, we often denote ω_{form} by ω; which of the two is meant should be clear from the context.

Notice that $[A,A]\cap A_0 = [A_0,A_0] = [R,R] = 0$ since $A_0 = R$ and R is commutative. In particular, we have $C_R^0(A)_0 = R$. The following result follows as in [25], by considering the Euler derivation associated with the $\mathbb{N}$-grading induced from A.

Proposition 4. *We have $H_R^0(A) = C_R^0(A)_0 = R$ and $H_R^n(A) = 0$ for all $n\geq 1$.*

[5] The notion of constant noncommutative symplectic form depends on the specific realization of A as a tensor algebra $T_R V$. In particular, this concept is not invariant under the R-superalgebra automorphism group $\mathrm{Aut_R(A)}$ (because a superalgebra automorphism need not preserve the $\mathbb{N}$-grading of A).

Using this in (3.6) gives a short exact sequence:

$$0 \to R \to C^0_R(A) \stackrel{\psi_\omega}{\to} \mathrm{Der}^\omega_l(A)[\tilde{\omega}] \to 0\,, \tag{4.3}$$

where surjectivity of ψ_ω follows by using $H^1_R(A) = 0$ in (3.5). In particular, *any relative symplectic derivation of A is hamiltonian.* The sequence (4.3) also shows that the hamiltonian of a symplectic derivation is determined up to addition of elements of R, which can be viewed as the subspace $C^0_R(A)_0$ of $C^0_R(A)$.

Any element f of $C^0_R(A)$ has a decomposition $f = \sum_{n\geq 0} f_n$, with $f_n \in C^0_R(A)_n$ (in particular, $f_0 \in C^0_R(A)_0 = R$). We say that f *has order k at zero* if $f_0 = f_1 = \ldots = f_{k-1} = 0$ and $f_k \neq 0$. We say that f *vanishes at zero* if $f_0 = 0$, i.e. f has order at least one at zero. We define the *canonical hamiltonian* of a symplectic derivation θ to be that hamiltonian of θ which vanishes at zero. The sequence (4.3) shows that the canonical hamiltonian exists and is unique.

4.2 Adapted bases and superquivers

Using the binary decomposition $E = \oplus_{(u,v)\in\mathcal{Q}_0\times\mathcal{Q}_0} E_{uv}$, consider a homogeneous basis (ψ_a) of the supervector space E having the following properties:

(1) $a = (u, v, j)$ is a multi-index with $u, v \in \mathcal{Q}_0$ and $j = 1 \ldots \dim_{\mathbb{C}} E_{uv}$

(2) ψ_{uvj} for $j = 1 \ldots \dim_{\mathbb{C}} E_{uv}$ is a homogeneous basis of E_{uv} for all $u, v \in \mathcal{Q}_0$.

We say that such a homogeneous basis is *adapted* to the binary decomposition of E. Setting $e_a := \Sigma\psi_a$, we let (s^a) be the basis of the super-vector space $V = E[1]^{\mathrm{v}}$ dual to (e_a):

$$s^a(e_b) = \delta^a_b\,.$$

Relation (2.2) shows that s^{uvj} are bases of $V_{uv} = (E[1]^{\mathrm{v}})_{uv} = (E_{uv}[1])^*$, odd dual to the bases ψ_{uvj} of E_{uv}. We set $\tilde{a} := \mathrm{deg} s^a = \mathrm{deg} e_a = [|a| + 1]$, where $|a| := |\psi_a|$.

It is convenient to keep track of indices by considering a quiver $\mathcal{Q}$ determined by the multi-indices a. Specifically, the *index quiver* $\mathcal{Q}$ is the quiver on the vertex set $\mathcal{Q}_0$ obtained by drawing an arrow from u to v for each $j = 1 \ldots \dim_{\mathbb{C}} V_{uv}$. With this construction, we can identify each multi-index a with the corresponding arrow of $\mathcal{Q}$. We let $\mathcal{Q}_1$ be the set of all arrows and $\mathcal{Q}_1(u, v)$ the subset of arrows going from u to v. We also let $h, t : \mathcal{Q}_1 \to \mathcal{Q}_0$ be the head and tail maps of $\mathcal{Q}$.

The index quiver is in fact a *superquiver*, being endowed with a map $\mathrm{deg} : \mathcal{Q}_1 \to \mathbb{Z}_2$ given by $\mathrm{deg}(a) = \tilde{a}$. An arrow a is called even if $\tilde{a} = 0$ and odd if $\tilde{a} = 1$. The path algebra $\mathbb{C}\mathcal{Q}$ becomes a superalgebra by declaring a path p to be even or odd if it contains an even or odd number of odd arrows. That is, we define the degree of p by the formula:

$$\tilde{p} := \sum_{j=1}^{k} \tilde{a}_j\,,$$

where $p = a_1 \ldots a_k$ is the arrow decomposition. The trivial paths are taken to be even. The path algebra is in fact $\mathbb{N} \times \mathbb{Z}_2$-graded, where the $\mathbb{N}$-grading is induced by the length of paths. We let $(\mathbb{C}\mathcal{Q})_n$ be the components of degree n with respect to the length grading.

JHEP11(2005)032

As usual, the subspace spanned by the trivial paths forms a finite-dimensional semisimple commutative algebra. We identify this with the boundary sector algebra R by sending the trivial path at u into the idempotent ϵ_u. On the other hand, the subspace spanned by the arrows is isomorphic with the supervector space V:

$$(\mathbb{C}\mathcal{Q})_1 \approx \oplus_{u,v\in\mathcal{Q}_0} \oplus_{a\in\mathcal{Q}_1(u,v)} \mathbb{C}[\tilde{a}] \approx V$$

via the identification $a \equiv s^a$. It is also clear that the R-superbimodule structure of V coincides with the $(\mathbb{C}\mathcal{Q})_0$-superbimodule structure induced on $(\mathbb{C}\mathcal{Q})_1$ by multiplication in the path algebra. In fact, the entire path algebra is isomorphic with the tensor algebra $T_R(\mathbb{C}\mathcal{Q})_1$ as an $\mathbb{N}\times\mathbb{Z}_2$-graded algebra. Combining these observations, we find an isomorphism of bigraded algebras:

$$A \approx \mathbb{C}\mathcal{Q}$$

which extends the isomorphism $V \approx (\mathbb{C}\mathcal{Q})_1$. Hence:

Choosing an adapted basis of E identifies the tensor algebra $A = T_R V$ with the path algebra of the index superquiver $\mathcal{Q}$.

In the next subsections, we explore the consequences of this identification.

4.3 Quiver description of the symplectic structure

Recall that the symplectic form ω on $E[1]$ corresponds to bilinear forms $\omega_{uv} : E[1]_{uv} \times E[1]_{vu} \to \mathbb{C}$ of common degree $\tilde{\omega}$, such that $\omega_{uv}(x,y) = (-1)^{\tilde{x}\tilde{y}+1}\omega_{vu}(y,x)$. We define coefficients $(\omega_{ab})_{a,b\in\mathcal{Q}_1}$ through:

$$\omega_{ab} = -\omega_{t(a)h(a)}(e_a, e_b) \in \mathbb{C} \qquad \text{if} \qquad t(a) = h(b) \qquad \text{and} \qquad h(a) = t(b)$$

and zero otherwise. The minus sign in this expression is introduced in order to simplify certain formulas which will appear in the next sections. In terms of the superbimodule symplectic form, we have $\omega(e_a, e_b) = -\omega_{ab}\epsilon_u$ where $u := t(a) = h(b)$. The coefficients have the graded-antisymmetry properties:

$$\omega_{ab} = (-1)^{\tilde{a}\tilde{b}+1}\omega_{ba} \tag{4.4}$$

and satisfy the selection rule:

$$\omega_{ab} = 0 \qquad \text{unless}\ \tilde{a} + \tilde{b} = \tilde{\omega}\,. \tag{4.5}$$

It is not hard to see that the inverse matrix $(\omega^{ab})_{a,b\in\mathcal{Q}_1}$ also satisfies the selection rule:

$$\omega^{ab} = 0 \qquad \textit{unless} \qquad \tilde{a} + \tilde{b} = \tilde{\omega}$$

and graded antisymmetry property:

$$\omega^{ab} = (-1)^{\tilde{a}\tilde{b}+1}\omega^{ba}\,.$$

Moreover, ω^{ab} vanishes unless $h(a) = t(b)$ and $t(a) = h(b)$.

Let $\hat{\omega}$ be the central element of $V \otimes_R V$ determined by ω. Since s^a is a vector space basis of V, we can expand $\hat{\omega} = \sum_{a,b} \hat{\omega}_{ab} s^a \otimes_R s^b = \sum_a s^a \otimes_R (s^a)'$, where we set $(s^a)' := \sum_b \hat{\omega}_{ab} s^b$. Then equation (4.1) gives $\omega(e_a, e_b) = \sum_c s^c(e_a(s^c)'(e_b)) = \hat{\omega}_{ab}$. Thus $\hat{\omega}_{ab} = -\omega_{ab}\epsilon_{t(a)}$ and we find that our definition of coefficients corresponds to the expansion:

$$\hat{\omega} = -\sum_{a,b \in \mathcal{Q}_1} \omega_{ab} s^a \otimes_R s^b \in V \otimes_R V\,,$$

where we used the relation $\epsilon_{t(a)} s^a = s^a$, which holds because $s^a \in V_{t(a)h(a)}$. Using equation (4.2), it follows that the non-commutative symplectic form induced on A is given by:

$$\omega_{\text{form}} = \frac{1}{2} \sum_{\substack{a,b \in \mathcal{Q}_1 \\ h(a)=t(b),h(b)=t(a)}} \omega_{ab} (ds^a ds^b)_c \equiv \frac{1}{2} \sum_{a,b \in Q_1} \omega_{ab} (dadb)_c\,. \tag{4.6}$$

In the last equality, notice that $(dadb)_c$ vanishes unless $h(a) = t(b)$ and $h(b) = t(a)$, which can be seen immediately by inserting idempotents ϵ_u in the appropriate places. As in [25, 26], it is easy to check that non-degeneracy of the constant two-form (4.6) amounts to non-degeneracy of the matrix $(\omega_{ab})_{a,b \in \mathcal{Q}_1}$. A *symplectic superquiver* is a superquiver whose path algebra is endowed with a constant symplectic form of type (4.6). Hence picking an adapted homogeneous basis allows us to encode the information of the symplectic superbimodule $(E[1], \omega)$ into a symplectic superquiver. Of course, the inverse correspondence also holds.

Recall that the topological metric ρ on E is given by $\omega = \rho \circ \Sigma^{\otimes 2}$. We define its coefficients trough:

$$\rho_{ab} = \rho(\psi_a, \psi_b)\,,$$

without a minus sign insertion. Equation $\omega(e_a, e_b) = (-1)^{\tilde{a}} \rho(\psi_a, \psi_b)$ gives:

$$\omega_{ab} = (-1)^{\tilde{a}+1} \rho_{ab}\,. \tag{4.7}$$

The coefficients of ρ have the properties:

$$\rho_{ab} = (-1)^{\tilde{a}\tilde{b}+\tilde{\omega}+1} \rho_{ba} = (-1)^{|a||b|} \rho_{ba}$$

and:

$$\rho_{ab} = 0 \text{ unless } \tilde{a} + \tilde{b} = \tilde{\omega} \Longleftrightarrow |a| + |b| = \tilde{\omega}\,.$$

Relation (4.7) shows that the inverse of the matrix (ρ_{ab}) takes the form:

$$\rho^{ab} = (-1)^{\tilde{b}+1} \omega^{ab}\,.$$

It is clear that the inverse matrix satisfies the relations:

$$\rho^{ab} = (-1)^{\tilde{a}\tilde{b}+\tilde{\omega}+1} \rho^{ba} = (-1)^{|a||b|} \rho^{ba}\,. \tag{4.8}$$

and

$$\rho^{ab} = 0 \quad \text{unless}\, \tilde{a} + \tilde{b} = \tilde{\omega} \Longleftrightarrow |a| + |b| = \tilde{\omega}\,. \tag{4.9}$$

The structure theorem for graded antisymmetric matrices implies the following:

(1) If $\tilde{\omega} = 0$, then we can find an adapted basis and an ordering $a_1, \dots, a_{2m}, a_{2m+1}, \dots, a_N$ of the arrows ($N = \mathrm{Card}\mathcal{Q}_1$) such that $a_1, \dots, a_{2m}$ are even, $a_{2m+1}, \dots, a_N$ are odd and:

$$\omega = \sum_{i=1}^{m} (da_i da_{i+m})_c + \frac{1}{2} \sum_{j=2m+1}^{N} (da_j da_j)_c \, .$$

Setting $p_i := a_i$ and $q_i = a_{i+m}$ for $i = 1 \dots m$ and $\xi_\alpha := a_{2m+\alpha}$ for $\alpha = 1 \dots N - 2m$, we can write this in the form:

$$\omega = (dp_i dq_i)_c + \frac{1}{2}(d\xi_\alpha d\xi_\alpha)_c \tag{4.10}$$

with even p_i, q_i and odd ξ_α. Hence $\omega_{p_i q_j} = -\omega_{q_j p_i} = \delta_{ij}$ and $\omega_{\xi_\alpha \xi_\beta} = \delta_{\alpha\beta}$.

(2) If $\tilde{\omega} = 1$, then $\mathrm{Card}\mathcal{Q}_1 = 2m$ for some integer m and we can find an adapted basis and an ordering $a_1, \dots, a_{2m}$ of the arrows such that $a_1, \dots, a_m$ are odd, $a_{m+1}, \dots, a_{2m}$ are even and:

$$\omega = \sum_{i=1}^{m} (da_i da_{i+m})_c \, . \tag{4.11}$$

Setting $p_i := a_i$ and $q_i = a_{i+m}$ for $i = 1 \dots m$, this becomes:

$$\omega = (dp_i dq_i)_c \tag{4.12}$$

with even p_i and odd q_i. Hence $\omega_{p_i q_j} = -\omega_{q_j p_i} = \delta_{ij}$.

In general, one can set $a^* = \sum_{b \in \mathcal{Q}_1} \omega_{ab} b$, which brings ω to the form $\omega = \frac{1}{2}(dada^*)_c$. For an even ω in the canonical basis (4.10), we have $p_i^* = q_i$, $q_i^* = -p_i$ and $\xi_\alpha^* = \xi_\alpha$. In this case, $*$ squares to minus the identity on the subspace spanned by q_i, p_i but to plus the identity on the subspace spanned by ξ_α. For odd ω in the basis (4.12), we have $p_i^* = q_i$ and $q_i^* = -p_i$, so $*$ squares to minus the identity on the entire subspace $A_1 = V$.

It is clear from the above that a D-brane system has different behavior depending on the parity of ω. We say that the system is *even* or *odd* if $\tilde{\omega} = 0$, respectively 1.

4.4 Quiver description of $C_R^0(A)$

Any element $f \in A$ has an expansion:

$$f = \sum_{p = \mathrm{path}} f_p p \tag{4.13}$$

where the sum is over all paths p of $\mathcal{Q}$ (including the trivial paths) and where $f_p \in \mathbb{C}$. In this and subsequent relations, we agree that only a finite number of coefficients are nonzero, so that all sums are finite. We can also write (4.13) as:

$$f = \sum_{n \geq 0} f_{a_1 \dots a_n} a_1 \dots a_n \, , \tag{4.14}$$

where we use implicit summation over the arrows a_j and we agree that $f_{a_1 \dots a_0} a_1 \dots a_0$ stands for the sum $c(f) := \sum_{u \in \mathcal{Q}_0} f_u \epsilon_u$ (with $f_u \in \mathbb{C}$). The product $a_1 \dots a_n$ vanishes unless it is a path. This is seen by inserting idempotents:

$$a_1 \dots a_i a_{i+1} \dots a_n = a_1 \dots a_i \epsilon_{h(a_i)} \epsilon_{t(a_{i+1})} a_{i+1} \dots a_n \, , \tag{4.15}$$

and noticing that the right hand side vanishes unless $h(a_i) = t(a_{i+1})$. Thus only the coefficients $f_{a_1 \dots a_n}$ which correspond to paths $a_1 \dots a_n$ are defined; for convenience, we define $f_{a_1 \dots a_n}$ to vanish if the word $a_1 \dots a_n$ is not a path.

The obvious relations

$$(a_1 \dots a_n)_c = (-1)^{(\tilde{a}_1 + \dots + \tilde{a}_i)(\tilde{a}_{i+1} + \dots + \tilde{a}_n)} (a_{i+1} \dots a_n a_1 \dots a_i)_c \,. \tag{4.16}$$

show that $(a_1 \dots a_n)_c$ vanishes unless $a_1 \dots a_n$ is a cycle of $\mathcal{Q}$ (as we will see below, it can still vanish even for a cycle). For a general element (4.14), relations (4.16) give:

$$f_c := \pi(f) = \sum_{n \geq 0} f_{a_1 \dots a_n} (a_1 \dots a_n)_c = \sum_{n \geq 0} f_{(a_1 \dots a_n)} (a_1 \dots a_n)_c \,,$$

where we introduced the 'cyclicized coefficients':

$$f_{(a_1 \dots a_n)} := \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{(\tilde{a}_1 + \dots + \tilde{a}_i)(\tilde{a}_{i+1} + \dots + \tilde{a}_n)} f_{a_{i+1} \dots a_n, a_1 \dots a_i} \,, \tag{4.17}$$

which satisfy $f_{(a_1 \dots a_n)} = (-1)^{(\tilde{a}_1 + \dots + \tilde{a}_i)(\tilde{a}_{i+1} + \dots + \tilde{a}_n)} f_{(a_{i+1} \dots a_n, a_1 \dots a_i)}$. For $n = 0$, we set $f_{(u)} = f_u$ for all $u \in \mathcal{Q}_0$.

The observations made above show that any element $f \in C^0_R(A)$ can be expanded as:

$$f = \sum_{n \geq 0} f_{a_1 \dots a_n} (a_1 \dots a_n)_c \,, \tag{4.18}$$

where the coefficients are taken to be graded cyclic:

$$f_{a_1 \dots a_n} = (-1)^{(\tilde{a}_1 + \dots + \tilde{a}_i)(\tilde{a}_{i+1} + \dots + \tilde{a}_n)} f_{a_{i+1} \dots a_n, a_1 \dots a_i}$$

and the term $n = 0$ in the sum stands for $\sum_{u \in \mathcal{Q}_0} f_u \epsilon_u \in R$. We also define the *strict coefficients* of $f \in C^0_R(A)$ by:

$$\bar{f}_{a_1 \dots a_n} := n f_{a_1 \dots a_n} \qquad \text{if}\, n \neq 0$$

and $\bar{f}_u := f_u$ for $n = 0$. Then the expansion of f_c becomes:

$$f_c = c(f) + \sum_{n \geq 1} \frac{\bar{f}_{a_1 \dots a_n}}{n} (a_1 \dots a_n)_c \,.$$

Consider the set $\mathcal{C}(\mathcal{Q})$ of cycles of $\mathcal{Q}$. We say that two cycles γ_1 and γ_2 are *equivalent*, and write $\gamma_1 \sim \gamma_2$, if they have the same length and differ by a cyclic permutation of their arrows (i.e. they differ only in the choice of the initial=terminal point of the cycle). This is an equivalence relation on $\mathcal{C}(\mathcal{Q})$, whose equivalence classes are known as *necklaces*. We let $N(\mathcal{Q}) := \mathcal{C}(\mathcal{Q})/\sim$ denote the set of necklaces, and write $[\gamma]$ for the equivalence class of a cycle γ. We define the length $l([\gamma])$ to be the length of any representative cycle. The $\mathbb{Z}_2$ degree of a necklace is the degree of any of its representatives. This gives a well-defined map from $N(\mathcal{Q})$ to $\mathbb{Z}_2$.

A cycle γ is called primitive if it cannot be written in the form $\gamma = u^k$ with u a non-trivial cycle and $k \geq 2$ (with this definition, the trivial paths ϵ_u are primitive). Any non-trivial cycle γ can be written uniquely in the form $\gamma = r(\gamma)^{p(\gamma)}$ where $p(\gamma) \in \mathbb{N}^*$ and $r(\gamma)$ is a primitive cycle. This representation is called the primitive decomposition of γ. The integer $p(\gamma)$ is called the period, while the path $r(\gamma)$ is called the primitive root of γ. Given a necklace ν and representatives $\gamma_1, \gamma_2 \in \nu$, we have $p(\gamma_1) = p(\gamma_2)$ and $r(\gamma_1) \sim r(\gamma_2)$. This allows us to define the period and primitive root of necklaces through $p([\gamma]) = p(\gamma)$ and $r([\gamma]) = [r(\gamma)]$.

Definition 3. *A* null necklace *is a necklace ν such that $p(\nu)$ is even and* $\deg r(\nu) = 1 \in \mathbb{Z}_2$.

Proposition 5. *Let γ be a cycle of the quiver $\mathcal{Q}$. The vector $\gamma_c = \pi(\gamma) \in C^0_R(A)$ vanishes if and only if the necklace $\nu := [\gamma]$ is null.*

We let $N_\bullet(\mathcal{Q}) := \{\nu \in N(\mathcal{Q}) | \nu \ \text{ is not null}\}$ be the set of non-null necklaces. Notice that the trivial paths ϵ_u are not null, and thus belong to $N_\bullet(\mathcal{Q})$. *Proof.* Let $\gamma = r^p$ be the primitive decomposition of γ. Commuting one copy of r to the right gives:

$$\gamma_c = \pi(r^p) = (-1)^{(p-1)\tilde{r}} \gamma_c \,,$$

where we noticed that $\deg(r)\deg(r^{p-1}) = \tilde{r}(p-1)$ in $\mathbb{Z}_2$. Thus $\gamma_c = 0$ if p is even and $\tilde{r}$ is odd. The converse follows from the description of π given in section 4.1 and the fact that r is the period of γ.

Consider a necklace $\nu = [a_1 \dots a_n]$. Relations (4.16) show that the subspace $V_\nu := \mathbb{C}(a_1 \dots a_n)_c \subset C^0_R(A)$ depends only on ν, while the proposition implies that $V_\nu = 0$ if ν is null and $V_\nu \approx \mathbb{C}$ otherwise. This gives a direct sum decomposition:

$$C^0_R(A) = \oplus_{\nu \in N_\bullet(\mathcal{Q})} V_\nu \,,$$

where $V_u := \mathbb{C}\epsilon_u$ for the null paths ϵ_u. Accordingly, any element $f \in C^0_R(A)$ decomposes as:

$$f = \sum_{\nu \in N_\bullet(\mathcal{Q})} f(\nu) \,,$$

where all but a finite number of the vectors $f(\nu) \in V_\nu$ vanish. If $l(\nu) = n > 0$, then:

$$f(\nu) = \sum_{a_1 a_2 \dots a_n \in \nu} f_{a_1 a_2 \dots a_n} (a_1 \dots a_n)_c \,,$$

where the coefficients are defined as in (4.18). Since $f_{a_1 \dots a_n}$ are cyclic, all vectors $f_{a_1 \dots a_n}$ $(a_1 \dots a_n)_c \in C^0_R(A)$ for which the cycle $a_1 \dots a_n$ belongs to a given necklace ν are equal. Thus:

$$f(\nu) = \bar{f}_{a_1 \dots a_n} (a_1 \dots a_n)_c \qquad \text{for any fixed cycle}\, a_1 \dots a_n \in \nu \,.$$

(in this relation, no summation over a_j is implied) .

JHEP11(2005)032

In view of these observations, the coefficients in the expansion (4.18) associated to null necklaces are not defined. For simplicity, we will define all such coefficients to be zero. Thus all coefficients associated with words $a_1 \dots a_n$ which fail to correspond to a cycle or correspond to a cycle in a null necklace are set to zero by definition. We will use this convention repeatedly in what follows.

Observation. $C^0_R(A)$ can be identified with the vector space $\mathbb{C}^{N_\mathcal{Q}} = \oplus_{\nu \in N_\mathcal{Q}} \mathbb{C}\nu$ generated by the set of non-null necklaces $N_\mathcal{Q}$. For this, pick an enumeration of $\mathcal{Q}_1$, let $c_\nu := \min \nu$ be the minimal representative of ν with respect to the induced lexicographic order on the set of paths of $\mathcal{Q}$, and identify V_ν with $\mathbb{C}$ by sending $\pi(c_\nu)$ into the complex unit. With this identification, we have $f(\nu) \equiv \bar{f}_{c_\nu}$ and $f \equiv \sum_{\nu \in N_\bullet(\mathcal{Q})} \bar{f}_{c_\nu} \nu$. Notice that such an identification requires that we pick an enumeration of $\mathcal{Q}_1$.

4.5 Cyclic derivatives and loop partial derivatives

The isomorphism $\Omega^1_R A \approx A \otimes V$ of section 4.1 shows that any one-form $w \in \Omega^1_R A$ has well-defined coefficients $w_a \in A$ determined by:

$$w = \sum_{a \in \mathcal{Q}_1} (w_a da)_c = \sum_{a \in \mathcal{Q}_1} (-1)^{\tilde{a}(\tilde{w}+1)} ((da) w_a)_c \, ,$$

where we noticed that $\tilde{w}_a = \tilde{w} + \tilde{a}$. Inserting idempotents shows that each $w_a \in A$ must be a linear combination of paths starting at $h(a)$ and ending at $t(a)$.

We define the *left and right quiver cyclic derivatives* $\overrightarrow{\delta}_a f, f \overleftarrow{\delta}_a \in A$ of an element $f \in C^0_R(A)$ through:

$$\bar{d} f = \sum_a ((f \overleftarrow{\delta}_a) da)_c = \sum_a (da(\overrightarrow{\delta}_a f))_c \, . \tag{4.19}$$

This gives linear maps $\overrightarrow{\delta}_a, \overleftarrow{\delta}_a : C^0_R(A) \to A$, satisfying $f \overleftarrow{\delta}_a = (-1)^{\tilde{a}(\tilde{f}+1)} \overrightarrow{\delta}_a f$. Equation (4.19) gives:

$$\begin{aligned} \overrightarrow{\delta}_a f &= \sum_n \bar{f}_{a a_1 \dots a_n} a_1 \dots a_n \\ f \overleftarrow{\delta}_a &= \sum_n \bar{f}_{a_1 \dots a_n a} a_1 \dots a_n \, . \end{aligned}$$

Notice that our convention (1.4) is crucial for these simple formulas. In particular, for any cycle $\gamma = a_1 \dots a_n$ of $\mathcal{Q}$:

$$\begin{aligned} \overrightarrow{\delta}_a (a_1 \dots a_n)_c &= \sum_{i=1}^n (-1)^{(\tilde{a}_1 + \dots + \tilde{a}_{i-1})(\tilde{a} + \tilde{a}_{i+1} + \dots + \tilde{a}_n)} \delta_{a, a_i} a_{i+1} \dots a_n a_1 \dots a_{i-1} \\ (a_1 \dots a_n)_c \overleftarrow{\delta}_a &= \sum_{i=1}^n (-1)^{(\tilde{a} + \tilde{a}_1 + \dots + \tilde{a}_{i-1})(\tilde{a}_{i+1} + \dots + \tilde{a}_n)} \delta_{a, a_i} a_{i+1} \dots a_n a_1 \dots a_{i-1} \, . \end{aligned} \tag{4.20}$$

These equations imply:

$$\begin{aligned} \pi(\overrightarrow{\delta}_a (a_1 \dots a_n)_c) &= \sum_{i=1}^n (-1)^{\tilde{a}(\tilde{a}_1 + \dots + \tilde{a}_{i-1})} \delta_{a, a_i} (a_1 \dots a_{i-1} a_{i+1} \dots a_n)_c \\ \pi((a_1 \dots a_n)_c \overleftarrow{\delta}_a) &= \sum_{i=1}^n (-1)^{\tilde{a}(\tilde{a}_{i+1} + \dots + \tilde{a}_n)} \delta_{a, a_i} (a_1 \dots a_{i-1} a_{i+1} \dots a_n)_c \, . \end{aligned} \tag{4.21}$$

It is clear that all terms in right hand side vanish unless a_i is a loop. Hence the linear operators $\pi \circ \overrightarrow{\delta}_a$ and $\pi \circ \overleftarrow{\delta}_a$ induced on $C^0_R(A)$ are non-trivial only when a is a loop.

We also define *loop partial derivatives* $\overrightarrow{\partial}_a, \overleftarrow{\partial}_a$ if a is a (non-trivial) *loop* of the quiver. Given such a loop of $\mathcal{Q}$, the loop derivatives $\overrightarrow{\partial}_a \in \mathrm{Der}_l(A)$ and $\overleftarrow{\partial}_a \in \mathrm{Der}_r(A)$ are the unique R-linear left and right derivations of A of degree $\tilde{a}$ such that:

$$\overrightarrow{\partial}_a b := b\overleftarrow{\partial}_a = \delta^b_a \epsilon_u \qquad \text{for all}\, b \in \mathcal{Q}_1\,, \tag{4.22}$$

where $u = h(a) = t(a)$. It is not hard to see that $\overrightarrow{\partial}_a$ and $\overleftarrow{\partial}_a$ are well-defined and $\overrightarrow{\partial}_a f = (-1)^{\tilde{a}(\tilde{f}+1)} f \overleftarrow{\partial}_a$. It is clear from (4.22) that loop partial derivatives supercommute:

$$[\overrightarrow{\partial}_a, \overrightarrow{\partial}_b] = [\overleftarrow{\partial}_a, \overleftarrow{\partial}_b] = 0 \qquad \text{for all loops}\, a, b \in \mathcal{Q}_1\,.$$

Let $\mathcal{Q}_1(u)$ be the subset of loops at the vertex u and let $a^* = \sum_{b\in\mathcal{Q}_1} \omega_{ab} b = \sum_{b\in\mathcal{Q}_1(u)} \omega_{ab} b \in A$ be the conjugate element of the loop a introduced in subsection 4.3. A simple computation gives the relation:

$$\bar{i}_{\overrightarrow{\partial}_a} \omega = \bar{d}\pi(a^*)\,,$$

which shows that $\overrightarrow{\partial}_a$ is the symplectic relative left derivation with hamiltonian $\pi(a^*) \in C^0_R(A)$. Similarly, one can view $\overleftarrow{\partial}_a$ as the symplectic relative right derivation with hamiltonian $\pi(a^*)$. Notice that the necklace of a loop a has only one representative, so one can identify loops with their projections through π; we will sometimes do so in what follows.

Let ∂^r_a and ∂^l_a be the complex-linear maps induced by $\overrightarrow{\partial}_a$ and $\overleftarrow{\partial}_a$ on $C^0_R(A) = A/[A, A]$. For any cycle $\gamma = a_1 \dots a_n$, we have:

$$\begin{aligned}
\partial^l_a \gamma_c &= \pi(\overrightarrow{\partial}_a \gamma) = \sum_{i=1}^n (-1)^{\tilde{a}(\tilde{a}_1+\ldots+\tilde{a}_{i-1})} \delta_{a,a_i} (a_1 \dots a_{i-1} a_{i+1} \dots a_n)_c \\
\gamma_c \partial^r_a &= \pi(\gamma \overleftarrow{\partial}_a) = \sum_{i=1}^n (-1)^{\tilde{a}(\tilde{a}_{i+1}+\ldots+\tilde{a}_n)} \delta_{a,a_i} (a_1 \dots a_{i-1} a_{i+1} \dots a_n)_c\,.
\end{aligned}$$

Comparing with (4.21) gives:

$$\partial^l_a = \pi \circ \overrightarrow{\delta}_a\,, \qquad \partial^r_a = \pi \circ \overleftarrow{\delta}_a\,.$$

Hence the cyclic and loop partial derivatives induce the same complex-linear operators on $C^0_R(A)$.

Given an element $x \in E[1]$, we expand $x = \sum_{a\in\mathcal{Q}_1} x^a e_a$ with $x^a \in \mathbb{C}$ and define the (relative) left cyclic derivative along x via:

$$\overrightarrow{\delta}_x f := \sum_{a\in\mathcal{Q}_1} x^a \overrightarrow{\delta}_a f \qquad \forall f \in C^0_R(A)\,.$$

If x is central in E, then x^a vanish unless a is a loop. In this case, we define the (relative) loop left partial derivative along x via:

$$\overrightarrow{\partial}_x f = \sum_{a=loop} x^a \overrightarrow{\partial}_a f\,.$$

JHEP11(2005)032

For a central element, these two notions induce the same map on $C^0_R(A)$:

$$\partial^l_x f = \sum_{a=loop} x^a \partial^l_a f\,.$$

These definitions are well-behaved with respect to changes of adapted bases. It is clear that $\partial^l_{e_a} = \partial^l_a$ etc.

Observations.

(1) Loop partial derivatives and quiver cyclic derivatives are related to certain double derivations introduced in [22]. Consider the A-superbimodule $A \otimes_{\mathbb{C}} A$, where we use the so-called outer superbimodule structure:

$$\alpha(a \otimes b)\beta := (\alpha a) \otimes (b\beta) \qquad \forall \alpha, \beta, a, b \in A\,.$$

Since R is a subalgebra of A, this is also an R-superbimodule. A relative double derivation of A is an R-linear derivation of the A-superbimodule $A \otimes A$. As in [22], consider the double left derivations determined by:

$$\mathbf{D}_a(b) = \epsilon_{t(a)} \otimes \epsilon_{h(a)} \delta^b_a \qquad \forall a, b \in \mathcal{Q}_1\,.$$

Then the loop partial derivatives can be recovered as:

$$\overrightarrow{\partial}_a = \mathbf{m} \circ \mathbf{D}_a \qquad \text{for } a \text{ a loop}$$

where $\mathbf{m} : A \otimes A \to A$ is the A-superbimodule morphism:

$$\mathbf{m}(a \otimes b) = ab\,.$$

Similarly, relations (4.20) show that the cyclic derivatives are induced by the maps $m \circ \mathbf{D}_a$, where $m : A \otimes A \to A$ is the linear map (*not* a bimodule morphism !):

$$m(a \otimes b) = (-1)^{\tilde{a}\tilde{b}} ba\,. \tag{4.23}$$

Indeed, it is not hard to see that $m \circ \mathbf{D}_a$ vanishes on $[A, A]$, so it induces a map from $C^0_R(A)$ to A which coincides with the cyclic derivative $\overrightarrow{\delta}_a$.

(2) Let $\mathcal{Q}$ be a quiver with a single vertex u. Then all arrows are loops and A is the free superalgebra $\mathbb{C}\langle\{a\}\rangle$ generated by these loops. In this case, $\overrightarrow{\partial}_a$ and $\overleftarrow{\partial}_a$ reduce to the standard left and right partial derivatives of the free algebra A. Moreover, the observations above show that $\overrightarrow{\delta}_a$ are a superized version of the objects considered in [19, 20]. This justifies our terminology.

4.6 Description of one-forms and closed two-forms

Consider the reduced tensor algebra $A_{\geq 1} := V \otimes_R A = T_R^{\geq 1} V = \oplus_{n\geq 1} V^{\otimes_R n}$, which coincides with the subspace of A spanned by all paths of length at least one. As in [25, 26] (see also [32]), it is easy to see that the super-vector space $C_R^1(A)$ is isomorphic with the center $A_{\geq 1}^R = (V \otimes_R A)^R$, the space spanned by the non-trivial cycles of the quiver. The isomorphism $\Xi : A_{\geq 1}^R \to C_R^1(A)$ has the form:

$$\sum_i x_i \otimes_R f_i \xrightarrow{\Xi} \sum_i (dx_i f_i)_c \in C_R^1(A) . \tag{4.24}$$

This follows from the observation that any element of $C_R^1(A)$ can be written uniquely as:

$$w = \sum_{n\geq 0} f_{aa_1\dots a_n} (daa_1 \dots a_n)_c = \sum_a (daf_a)_c = \Xi(\sum_a af_a) , \tag{4.25}$$

where $f_a := \sum_{n\geq 0} f_{aa_1\dots a_n} a_1 \dots a_n$ and the complex coefficients $f_{a_1\dots a_n}$ are taken to vanish unless $a_1 \dots a_n$ is a cycle. Then $\Xi^{-1}(w) = \sum_{n\geq 0} f_{a_1\dots a_n} a_1 \dots a_n := f = \sum_a af_a$. For $g = \sum_{n\geq 0} g_{a_1\dots a_n} a_1 \dots a_n \in A_{\geq 1}^R$, we have $\bar{d}\pi(g) = \bar{d}g_c = (da \overrightarrow{\delta}_a g_c)_c = \sum_{n\geq 0} \bar{g}_{(aa_1\dots a_n)}$ $(daa_1 \dots a_n)_c$, where $\bar{g}_{(a_1\dots a_n)} := ng_{(a_1\dots a_n)}$ are the strict coefficients of g_c. Thus $w = \bar{d}(g)_c$ iff $f_{a_1\dots a_n} = \bar{g}_{(a_1\dots a_n)}$. It is clear from these observations that w is exact iff its coefficients $f_{a_1\dots a_n}$ are graded-cyclic; in this case, we can take $g = \frac{1}{n} f_{a_1\dots a_n} a_1 \dots a_n$ and we have $f_a = \overrightarrow{\delta}_a g$.

Thus $\Xi(f) \in C_R^1(A)_{\rm closed} = C_R^1(A)_{\rm exact}$. We let $A_{\geq 1}^{\rm c}$ be the *cyclic subspace* of $A_{\geq 1}^R$, i.e. the subspace consisting of elements with graded-cyclic coefficients. The space $A_{\geq 1}^{\rm c}$ consists of linear combinations of non-trivial cycles, such that cycles belonging to the same necklace of length n appear with coefficients related by the action of $\mathbb{Z}_n$. This subspace provides an embedding of $C_R^0(A)/R$ into $A_{\geq 1}$. In fact, the projection $\pi : A \to C_R^0(A)$ induces an isomorphism $A_{\geq 1}^{\rm c} \approx C_R^0(A)/R = A_{\geq 1}/[A, A]$ (recall that $[A, A]_0 = [R, R] = 0$ so $[A, A] \subset A_{\geq 1}$). Thus we have a vector space decomposition $A_{\geq 1} = [A, A] \oplus A_{\geq 1}^{\rm c}$, as well as $A_{\geq 1}^R = [A, A]^R \oplus A_{\geq 1}^{\rm c}$. In particular, the subspace $A_{\geq 1}^{\rm c}$ gives a natural complement of $[A, A]^R$ inside $A_{\geq 1}^R$.

Since the Karoubi complex is acyclic in positive degrees, we have $C_R^2(A)_{\rm closed} = \bar{d}(C_R^1(A))$ and the isomorphism Ξ shows that any closed two-form can be written as:

$$u = -\sum_{n\geq 0} (dadf_a)_c = \bar{d}\Xi(f) . \tag{4.26}$$

where $f = af_a \in A_{\geq 1}^R$ is a combination of cycles of length at least one. Since $\Xi(A_{\geq 1}^{\rm c}) = C_R^1(A)_{\rm closed}$, the two-form (4.26) vanishes precisely when f belongs to the cyclic subspace $A_{\geq 1}^{\rm c}$. Thus $\ker(d\Xi) = A_{\geq 1}^{\rm c}$, and the map $d\Xi$ induces an isomorphism:

$$\kappa : [A, A]^R \xrightarrow{\approx} C_R^2(A)_{\rm closed} \tag{4.27}$$

between the complement $[A, A]^R$ of this subspace in $A_{\geq 1}^R$ and the space of closed two forms.[6]

[6]This result is also discussed in [61]. I thank V. Ginzburg for pointing this out.

4.7 Quiver description of the Kontsevich bracket

Consider the non-commutative symplectic form (4.6) on A. For $\theta \in \mathrm{Der}_l(A)$, we set $\theta(a) := \theta^a \in A$. R-linearity of θ implies that θ^a is a linear combination of paths which start at $t(a)$ and end at $h(a)$. Equation (4.19) gives:

$$\bar{L}_\theta f = \bar{i}_\theta \bar{d} f = \pi \left(i_\theta \sum_{a\in\mathcal{Q}_1} (da) \overrightarrow{\delta}_a f \right) = \sum_{a\in\mathcal{Q}_1} (\theta(a) \overrightarrow{\delta}_a f)_c \,.$$

Thus:

$$\bar{L}_\theta f = \sum_{a\in\mathcal{Q}_1} (\theta^a \overrightarrow{\delta}_a f)_c \qquad \forall \theta \in \mathrm{Der}_l(A) \,. \tag{4.28}$$

If θ is homogeneous of degree $\tilde{\theta}$, we set:

$$\theta_a := \theta^b \omega_{ba} \,.$$

Notice that $\tilde{\theta_a} = \tilde{a} + \tilde{\omega} + \tilde{\theta}$. Expanding $\theta^a = \sum_{n\geq 0} \theta_{a_1\dots a_n}{}^a a_1 \dots a_n$ with $\theta_{a_1\dots a_n}{}^a \in \mathbb{C}$, we find $\theta_a = \sum_{n\geq 0} \theta_{a_1\dots a_n a} a_1 \dots a_n$, where:

$$\theta_{a_1\dots a_n a} = \theta_{a_1\dots a_n}{}^b \omega_{ba} \,.$$

As usual, $\theta_{a_1\dots a_n}{}^a$ are taken to vanish unless $a_1 \dots a_n$ is a path. Also notice that $\theta_{a_1\dots a_n}{}^a$ vanishes automatically unless this path starts at $t(a)$ and ends at $h(a)$. Similarly, $\theta_{a_1\dots a_n}$ vanishes unless $a_1 \dots a_n$ is a cycle of $\mathcal{Q}$.

An easy computation gives:

$$\bar{i}_\theta \omega = (\theta_a da)_c = \sum_{n\geq 0} \theta_{a_1\dots a_n a} (a_1 \dots a_n da)_c \,. \tag{4.29}$$

Given $f \in C^0_R(A)$, we have $\bar{d} f = (f \overleftarrow{\delta}_a da)_c$. Comparing with (4.29) gives $(\theta_f)_a = f \overleftarrow{\delta}_a$, where θ_f is the hamiltonian vector field of f. Hence the map $\psi_\omega : C^0_R(A) \to \mathrm{Der}_l(A)$ of (4.3) is given by:

$$\theta_f(a) := \theta^a_f = f \overleftarrow{\delta}_b \omega^{ba} = \sum_{n\geq 0} \bar{f}_{a_1\dots a_n b} \omega^{ba} s^{a_1} \dots s^{a_n} \,, \tag{4.30}$$

where $\bar{f}_{a_1\dots a_n}$ are the strict coefficients of f. This allows us to write the Kontsevich bracket in more familiar form.

Proposition 6. *We have* $\{f, g\} = (f \overleftarrow{\delta}_a \omega^{ab} \overrightarrow{\delta}_b g)_c$ *for all* $f, g \in C^0_R(A)$.

Proof.

Using (3.8), (4.28) and (4.30), we compute $\{f, g\} = \bar{L}_{\theta_f}(g) = (\theta^a_f \overrightarrow{\delta}_a g)_c = (f \overleftarrow{\delta}_b \omega^{ba} \overrightarrow{\delta}_a g)_c$.

4.8 Some canonical forms and coefficient expressions

In this subsection, we give some expressions which are useful in applications. Let $W \in C^0_R(A)$ be an element of degree $\tilde{\omega}+1$. As we will see in the next section, the boundary generating function of a topological D-brane system is such an element.

For $\tilde{\omega}=0$, let us choose an adapted basis as in (4.10). Then W is odd, and one finds:

$$\begin{aligned}\{W,W\} &= (W\overleftarrow{\delta}_{p_i}\overrightarrow{\delta}_{q_i}W - W\overleftarrow{\delta}_{q_i}\overrightarrow{\delta}_{p_i}W)_c + (W\overleftarrow{\delta}_{\xi_\alpha}\overrightarrow{\delta}_{\xi_\alpha}W)_c \\ &= 2(W\overleftarrow{\delta}_{p_i}\overrightarrow{\delta}_{q_i}W)_c + (W\overleftarrow{\delta}_{\xi_\alpha}\overrightarrow{\delta}_{\xi_\alpha}W)_c\end{aligned}$$

since $(W\overleftarrow{\delta}_{p_i}\overrightarrow{\delta}_{q_i}W)_c = -(W\overleftarrow{\delta}_{q_i}\overrightarrow{\delta}_{p_i}W)_c$. Also notice that $W\overleftarrow{\delta}_{\xi_\alpha} = \overrightarrow{\delta}_{\xi_\alpha}W$.

Now let $\tilde{\omega}=1$ and choose an adapted basis as in (4.12). Then W is even and we have:

$$\{W,W\} = (W\overleftarrow{\delta}_{p_i}\overrightarrow{\delta}_{q_i}W - W\overleftarrow{\delta}_{q_i}\overrightarrow{\delta}_{p_i}W)_c = 2(W\overleftarrow{\delta}_{p_i}\overrightarrow{\delta}_{q_i}W)_c\,, \tag{4.31}$$

since again $(W\overleftarrow{\delta}_{p_i}\overrightarrow{\delta}_{q_i}W)_c = -(W\overleftarrow{\delta}_{q_i}\overrightarrow{\delta}_{p_i}W)_c$.

One can also extract the coefficient expression of the cyclic bracket by direct computation. The case relevant for us is as follows. For W as above, notice that ρ^{ab} can be used to raise and lower indices 'from the left':

$$\begin{aligned}\bar{W}_{a_1\dots a_{i-1}}{}^{a}{}_{a_{i+1}\dots a_n} &:= \rho^{ab}\bar{W}_{a_1\dots a_{i-1}ba_{i+1}\dots a_n}\,,\\ \bar{W}_{a_1\dots a_{i-1}ba_{i+1}\dots a_n} &:= \rho_{ab}\bar{W}_{a_1\dots a_{i-1}}{}^{b}{}_{a_{i+1}\dots a_n}\,.\end{aligned}$$

Then it is shown in appendix D that that bracket of W with itself takes the form:

$$\begin{aligned}\frac{1}{2}\{W,W\} = \frac{1}{2}\bar{W}_a\bar{W}^a + \sum_{n\geq 1}\frac{1}{n}\left(\sum_{0\leq i+j\leq n}(-1)^{\tilde{a}_1+\dots+\tilde{a}_i}\bar{W}_{a_1\dots a_i a a_{i+j+1}\dots a_n}\bar{W}^{a}{}_{a_{i+1}\dots a_{i+j}}\right)\times\\ \times(a_1\dots a_n)_c\,,\end{aligned}$$

which is valid irrespective of the degree of ω.

5. Geometry of finite D-brane systems

Consider a finite topological D-brane system with total boundary space E and boundary algebra R. As before, we let $V=E[1]^{\mathrm{v}}$ and consider the tensor algebra $A=T_RV$. As explained in section 2, the data of all integrated boundary correlators on the disk is encoded by an R-superbimodule structure on E, together with a cyclic and unital weak A_∞ structure on this superbimodule. We will use the machinery developed in the previous two sections to encode this into a 'noncommutative generating function' $W\in C^0_R(A)$ subject to simple constraints. To this end, we pick an adapted basis of E and let $\mathcal{Q}$ be its index superquiver.

5.1 Geometric description of cyclic weak A_∞ structures

It turns out that a weak A_∞ structure on the R-superbimodule E is the same as an *odd* relative derivation Q of the tensor algebra A. With our conventions, the relation is as follows. Picking adapted coordinates, we define the coefficients of Q through:

$$Q(a) = \sum_{n\geq 0} Q_{a_1\dots a_n}{}^{a} a_1\dots a_n \tag{5.1}$$

and construct odd linear maps $r_n : E[1]^{\otimes n} \to E[1]$ via:

$$r(e_{a_1} \dots e_{a_n}) = Q_{a_1 \dots a_n}{}^a e_a \, . \tag{5.2}$$

Thus $Q(s^a) = \sum_{n\geq 0} s^a(r(e_{a_1} \dots e_{a_n}))s^{a_1} \otimes_R \dots \otimes_R s^{a_n}$. Thinking in terms of arrows, it is clear that r_n are R-multilinear. Since Q is odd, we have $[Q,Q] = 2Q^2$, which implies that Q^2 is a derivation of A. Since a generate the algebra, this means that the condition $Q^2 = 0$ is equivalent with $Q^2(a) = 0$ for all a. Using expansion (5.1), one finds that this amounts to the relations:

$$\sum_{0\leq i+j\leq n} (-1)^{\tilde{a}_1+\dots+\tilde{a}_i} Q_{a_1\dots a_i b a_{i+j+1}\dots a_n}{}^a Q_{\ a_{i+1}\dots a_{i+j}}{}^b = 0 \qquad \text{for all}\, n \geq 0\,,$$

which are the A_∞ constraints (2.4).

It is also not hard to check that the nilpotent derivation Q is symplectic iff the associated A_∞ structure is cyclic. An easy way to see this is as follows. By the exact sequence (4.3), we have that Q is symplectic iff it is hamiltonian, which via equation (4.30) amounts to the existence of a $W \in C^0_R(A)$ such that:

$$Q_{a_1\dots a_n}{}^a = \bar{W}_{a_1\dots a_n b}\omega^{ba} \tag{5.3}$$

or, equivalently:

$$Q_{a_1\dots a_n} = \bar{W}_{a_1\dots a_n} \, . \tag{5.4}$$

As usual, we have set $Q_{a_1\dots a_n a} := Q_{a_1\dots a_n}{}^b\omega_{ba}$. It is clear that a W exists if and only if the coefficients $Q_{a_1\dots a_n}$ are cyclic. We have:

$$\begin{aligned} \rho(e_{a_0}, r_n(e_{a_1}\dots e_{a_n})) &= (-1)^{\tilde{a}_0}\omega(e_{a_0}, r_n(e_{a_1}\dots e_{a_n})) = (-1)^{1+\tilde{a}_0\tilde{\omega}}\omega(r_n(e_{a_1}\dots e_{a_n}), e_{a_0}) \\ &= (-1)^{\tilde{a}_0\tilde{\omega}}\bar{W}_{a_1\dots a_n a_0} = \bar{W}_{a_0\dots a_n} \, , \end{aligned} \tag{5.5}$$

where we used the superselection rules for ω and W. Thus:

$$\rho(e_{a_0}, r_n(e_{a_1}\dots e_{a_n})) = \bar{W}_{a_0 a_1\dots a_n} \, , \tag{5.6}$$

and we see that $L_Q\omega = 0$ implies that the left hand side is cyclic, which is the cyclicity constraint (2.7). Conversely, if the LHS is cyclic then we define W through equation (5.6). Then relations (5.5) show that $Q_{a_1\dots a_n a_0} = \bar{W}_{a_1\dots a_n a_0}$, i.e. Q is symplectic with hamiltonian W. Combining everything and noticing that $\tilde{W} = \tilde{\omega} + 1$ (because Q is odd), we have:

Giving a cyclic weak A_∞ structure on E amounts to giving an element $W \in C^0_R(A)$, of degree $\tilde{\omega} + 1$, such that $\{W, W\} = 0$.

Observation. The triplet $(A, Q, \omega_{\text{form}})$ can be viewed as a noncommutative generalization of the so-called QP-manifolds of [30], while the doublet (A, Q) generalizes the concept of Q-manifold discussed in the same paper (notice, though, that we consider both even and odd symplectic forms, so we generalize the work of [30] in two directions). It was shown in [30] that QP-manifolds give the general geometric setting of the classical BV-formalism. Accordingly, for odd symplectic forms, the triplet $(A, Q, \omega_{\text{form}})$ defines a noncommutative version of that formalism.

5.2 The unitality constraint

We saw that a cyclic weak A_∞ structure on E is the same as an R-linear symplectic derivation $Q \in \mathrm{Der}_l^\omega(A)$ such that $Q^2 = 0$. We let W be the *canonical* hamiltonian of Q, i.e. that hamiltonian which vanishes at zero (see section 4.1). Explicitly, equations (5.4) and (5.6) give:

$$W = \sum_{n\geq 0} \frac{1}{n+1} \rho(e_{a_0}, r_n(e_{a_1} \dots e_{a_n}))(s^{a_0} \dots s^{a_n})_c\,, \tag{5.7}$$

which allows us to reconstruct r_n from W provided that we know $\omega = \rho \circ \Sigma^{\otimes 2}$. The homological derivation Q can be recovered as the hamiltonian derivation defined by W, which amounts to relations (5.4).

For a topological D-brane system, the underlying weak A_∞ structure should be unital. To formulate this condition in terms of W, we write the unitality constraints (2.5) as:

$$\begin{aligned} r_n(e_{a_1} \dots e_{a_{j-1}}, \lambda, e_{a_{j+1}} \dots e_{a_n}) &= 0 \qquad \text{for all}\, n \neq 2 \quad \text{and all}\, j = 1 \dots n \\ -r_2(\lambda, e_a) = (-1)^{\tilde{a}} r_2(e_a, \lambda) &= e_a\,, \end{aligned} \tag{5.8}$$

where λ is an odd central element of E. Given $\lambda = \oplus_{u\in\mathcal{Q}_0} \lambda_u \in E[1]^R$ with $\lambda_u \in E_{uu}[1]$, we can choose adapted coordinates such that each λ_u is one of the odd basis elements $\{e_a\}$. We then let $\sigma_u \in E[1]^{\mathrm{v}}$ be the corresponding elements of the dual basis $\{s^a\}$ of V (those dual basis elements which satisfy $s^{\sigma_u}(e_a) = \delta_{e_a, \lambda_u}$). It is clear that each σ_u is an odd loop of the quiver starting and ending at the vertex u. With such a choice of adapted basis, we have $\lambda_u = e_{\sigma_u}$ and the first row in (5.8) is equivalent with:

$$W_{a_1 \dots a_n} = 0 \text{ if } n \neq 3 \quad \text{and any of the arrows}\, a_j \quad coincideswithanyoftheloops\ \sigma_u \tag{5.9}$$

while the second row amounts to:

$$\bar{W}_{\sigma_u ab} = -\omega_{ab} \Longleftrightarrow W_{\sigma_u ab} = -\frac{1}{3}\omega_{ab} \qquad \text{for}\, t(a) = h(b) = u \qquad \text{and} \qquad h(a) = t(b)\,. \tag{5.10}$$

Hence unitality of (r_n) boils down to the requirement that the adapted basis $\{\psi_a\}$ can be chosen such that the vertices the index quiver carry distinguished odd loops σ_u satisfying (5.9) and (5.10).

The two conditions above say that W takes the form:

$$W = W_g + W_d \tag{5.11}$$

where the 'generic' contribution is given by:

$$W_g := -\omega_{ab}(\sigma ab)_c = - \sum_{\substack{a,b\in\mathcal{Q}_1, u\in\mathcal{Q}_0 \\ t(a)=h(b)=u\, h(a)=t(b)}} \omega_{ab}(\sigma_u ab)_c \tag{5.12}$$

while the 'deformation part' W_d vanishes at zero and is independent of all σ_u. In the first form of the last expression, we used Einstein summation over a and b and have set $\sigma = \sum_{u\in\mathcal{Q}_0} \sigma_u \in A_1$. Notice the lack of a $1/3$ prefactor in (5.12); this is because we brought all terms to a form in which a σ_u insertion appears in the first position.

To describe this more elegantly, notice[7] that (5.11) together with (5.12) amounts to the condition:

$$\overrightarrow{\delta}_\lambda W = -\sum_{a,b\in\mathcal{Q}_1}\omega_{ab}ab$$

where $\lambda := \sum_{u\in\mathcal{Q}_0}\lambda_u$ and $\overrightarrow{\delta}_\lambda := \sum_{u\in\mathcal{Q}_0}\overrightarrow{\delta}_{\sigma_u}$ as in subsection 4.5. Using the graded antisymmetry of ω_{ab}, the last relation takes the form:

$$\overrightarrow{\delta}_\lambda W = -\frac{1}{2}\sum_{a\in\mathcal{Q}_1}[a,a^*] \in [A,A]_2^R\,, \tag{5.13}$$

where we introduced the conjugate variables

$$a^* := \sum_{b\in\mathcal{Q}_1}\omega_{ab}b = \sum_{\substack{b\in\mathcal{Q}_1\\ h(a)=t(b),h(b)=t(a)}}\omega_{ab}b \tag{5.14}$$

as in subsection 4.3. Here $[A,A]_2$ is the subspace of $[A,A]$ consisting of elements of degree two with respect to the $\mathbb{N}$-grading, while $[A,A]_2^R\subset[A,A]^R$ is the centralizer of $[A,A]_2$ in R.

Relation (5.13) allows one reconstruct ω from W. To formulate this invariantly, remember from subsection 4.6 that the space of closed noncommutative two-forms $C^2_R(A)_{\rm closed}$ is isomorphic with $[A,A]^R$. Restricting the map (4.27) to the subspace $C^2_R(A)_2\subset C^2_R(A)_{\rm closed}$ of constant two-forms gives an isomorphism $[A,A]_2^R \stackrel{\approx}{\rightarrow} C^2_R(A)_2$ whose explicit form is given by relation (4.26):

$$\sum_{a,b\in\mathcal{Q}_1} f_{ab}ab \stackrel{\kappa}{\rightarrow} -\sum_{a,b\in\mathcal{Q}_1} f_{ab}(dadb)_c\,. \tag{5.15}$$

Here $f=\sum_{a,b}f_{ab}ab$ is the general element of $[A,A]_2^R$, with graded-antisymmetric complex coefficients $f_{ab}=(-1)^{1+\tilde{a}\tilde{b}}f_{ba}$, so we can also write $f=\frac{1}{2}\sum_{a,b}f_{ab}[a,b]$. Applying this to the noncommutative symplectic form, we find:

$$\kappa^{-1}(\omega) = -\frac{1}{2}\omega_{ab}ab = -\frac{1}{4}\sum_{a\in\mathcal{Q}_1}[a,a^*]\,. \tag{5.16}$$

Thus relation (5.13) can be written as either of the following equivalent conditions:

$$\overrightarrow{\delta}_\lambda W = 2\kappa^{-1}(\omega) \Leftrightarrow \omega = \frac{1}{2}\kappa(\overrightarrow{\delta}_\lambda W)\,. \tag{5.17}$$

These observations allow us to write the unitality constraint (5.13) as condition (5.17). In particular, the element $\overrightarrow{\delta}_\lambda W\in[A,A]_2^R$ must belong to the subspace spanned by the κ-preimages of quiver symplectic forms. To describe this space, notice that any element $\mu\in[A,A]_2^R$ can be expanded uniquely as:

$$\mu = -\frac{1}{4}\sum_{a\in\mathcal{Q}_1}[a,a^*]\,, \tag{5.18}$$

[7] For an element $f=\sum_{n\geq 0}f_{a_1\ldots a_n}(a_1\ldots a_n)_c\in C^0_R(A)$, the condition $\overrightarrow{\delta}_a f=0$ amounts to vanishing of all cyclic coefficients $f_{a_1\ldots a_n}$ for which one of the a_j coincides with a. Further, the cyclic derivative $\overrightarrow{\delta}_\lambda W$ determines $\overrightarrow{\delta}_{\sigma_u}W=\epsilon_u\overrightarrow{\delta}_\lambda W$. Thus (5.11) and (5.12) amount to $\overrightarrow{\delta}_\lambda(W+\omega_{ab}(\sigma ab)_c)=0$, which gives the desired statement.

where each a^* is a linear combination of arrows going from $h(a)$ to $t(a)$. We say that μ is *non-degenerate* if the elements $(a^*)_{a\in\mathcal{Q}_1}$ form a basis of V; in this case, we can expand a^* as in (5.14), with coefficients ω_{ab} forming the entries of a graded-antisymmetric non-degenerate matrix. Moreover, it is clear that μ has $\mathbb{Z}_2$-degree $\tilde{\omega}$ iff this matrix satisfies the selection rules (4.5). We let $\mathrm{Mom_V} \subset [\mathrm{A},\mathrm{A}]_2^{\mathrm{R}}$ be the $\mathbb{Z}_2$-homogeneous subspace of non-degenerate elements in $[A,A]_2^R$ and let $\mathrm{Mom}_\mathrm{V}^0$ and $\mathrm{Mom}_\mathrm{V}^1$ be its homogeneous components. The observations made above show that κ induces an isomorphism between the space $CNS^{\tilde{\omega}}(V)$ of constant non commutative symplectic forms on A having degree $\tilde{\omega}$ and the space $\mathrm{Mom}_\mathrm{V}^{\tilde{\omega}}$. It follows from [21, proposition 8.1.1] that Mom_V is the space of noncommutative moment maps associated to quiver symplectic forms on the path algebra A.

We can now formulate the unitality criterion as follows:

Proposition 7. *Let $E[1]$ be a symplectic R-superbimodule of finite complex dimension whose symplectic form has degree $\tilde{\omega}$, let $A := T_R E[1]^{\mathrm{v}}$ and let W be an element of $C_R^0(A)$ which has degree $\tilde{\omega}+1$ and vanishes at zero. Let ω be the noncommutative symplectic form induced on A and assume that $\{W,W\}=0$, where $\{.,.\}$ is the Kontsevich bracket defined by ω. Then the following statements are equivalent:*

(1) The cyclic weak A_∞ structure determined by W on $E[1]$ is unital

(2) There exists an odd central element $\lambda \in E^R$ such that $\frac{1}{2}\overrightarrow{\delta}_\lambda W = \kappa^{-1}(\omega)$.

In this case:

(a) $\Sigma\lambda$ is the unit of the A_∞ structure.

(b) The element $\mu := \frac{1}{2}\overrightarrow{\delta}_\lambda W \in A$ belongs to the subspace $\mathrm{Mom_V}$ of A

(c) The non-commutative symplectic form can be recovered via the relation $\omega = \kappa(\mu)$.

(d) Let $\lambda = \sum_{u\in\mathcal{Q}_0}\lambda_u$ ($\lambda_u \in E_{uu}$) be the decomposition of λ, and choose an adapted basis e_a of $E[1]$ containing λ_u among the basis elements. Let $s^a \equiv a$ be the dual basis, and let σ_u its elements associated with λ_u. Then σ_u correspond to odd loops of the associated superquiver and W takes the form given in eqs. (5.11) and (5.12), where W_d vanishes at zero and is independent of all σ_u.

5.3 Noncommutative geometry of D-brane systems

Combining the discussion of the previous subsections, we have the following non-commutative geometric description of finite topological D-brane systems:

Let R be a finite-dimensional semisimple commutative algebra over $\mathbb{C}$ and E an R-superbimodule which is finite-dimensional over $\mathbb{C}$. Giving a finite topological D-brane system with boundary decomposition described by (R, E) and topological metrics of $\mathbb{Z}_2$-degree $\tilde{\omega}$ amounts to giving a 'noncommutative function' $W \in C_R^0(A)$ on the tensor algebra $A = T_R E[1]^{\mathrm{v}}$ and an odd central element $\lambda \in E^R$ such that the following conditions are satisfied:

JHEP11(2005)032

(1) *W vanishes at zero and has $\mathbb{Z}_2$-degree $\tilde{\omega}+1$.*

(2) *The element $\mu := \frac{1}{2}\overrightarrow{\delta}_\lambda W$ belongs to $\mathrm{Mom}_{\mathrm{V}}^{\tilde{\omega}}$*

(3) *We have $\{W, W\} = 0$, where $\{.,.\}$ is the Kontsevich bracket determined on $C_R^0(A)$ by the constant noncommutative symplectic form $\omega := \kappa(\mu)$.*

Observations.

(1) The coefficient expressions given in section 4.8 show that equation $\{W, W\} = 0$ is equivalent with:

$$\sum_{0\leq i+j\leq n}(-1)^{\tilde{a}_1+\dots+\tilde{a}_i}\bar{W}_{a_1\dots a_i a a_{i+j+1}\dots a_n}\bar{W}^a{}_{a_{i+1}\dots a_{i+j}} = 0 \qquad \text{for all}\, n\geq 0\,, \tag{5.19}$$

where lifting of indices is done from the left with $\rho^{ab} = (-1)^{\tilde{b}+1}\omega^{ab}$, and ω^{ab} is the inverse of the matrix $\omega_{ab} = -\bar{W}_{\sigma ab}$. The first equation (for $n = 0$) is $\bar{W}_a\bar{W}^a = 0$. Notice that ω is determined by W, so the equations are not quadratic. This countable system of nonlinear algebraic conditions is a non-commutative analogue of the WDVV equations [3].

(2) It was shown in [7] that the background satisfies the string equations of motion iff the underlying A_∞ algebra is minimal. It is clear that this amounts to the requirement that W has order at least 3 at the origin.

(3) The structure given above can be viewed as an 'off-shell extension' of the 'boundary part' of the data described in [23, 39, 40]. The latter arises in the particular case when W has degree three at the origin (i.e. the underlying A_∞ structure is minimal), and can be recovered by forgetting all terms of W of order higher than 3. In physics language, the structure of [23, 39, 40] corresponds to keeping only the boundary three-point functions on the disk, thereby forgetting all *integrated* amplitudes. As explained in the introduction to [23], this reflects the difference between two-dimensional topological field theory and topological string theory, namely in the topological field theory one does not consider integration of amplitudes over the moduli space of the underlying Riemann surface (since by definition the worldsheet metric is not a dynamical variable).

5.4 Deformations of the underlying string theory

Each solution of the constraints described in the previous subsection represents the tree-level boundary data of an open topological string theory. We would like to make some basic observations about the space of such theories.

It is instructive to consider the trivial approximation $W_d = 0$ i.e. $W = W_g$, which — as we shall see in a moment — is appropriate under certain assumptions. Starting from $W_g = -\omega_{ab}(\sigma ab)_c$, we compute:

$$-W_g\overleftarrow{\delta}_a = \omega_{\alpha a}\sigma\alpha + (-1)^{\tilde{\beta}\tilde{\omega}}\omega_{a\beta}\beta\sigma + (-1)^{\tilde{\omega}}\delta_a^\sigma\omega_{\alpha\beta}\alpha\beta \tag{5.20}$$

and:

$$-\,\omega^{ab}\overrightarrow{\delta}_b W_g = (-1)^{\tilde{\omega}} a\sigma + (-1)^{\tilde{a}+\tilde{\omega}}\sigma a + \omega^{a\sigma}\omega_{\gamma\delta}\gamma\delta\,. \qquad (5.21)$$

Combining these equations and using appropriate cyclic permutations gives:

$$\{W_g, W_g\}_c = (W_g \overleftarrow{\delta}_a \omega^{ab}\overrightarrow{\delta}_b W_g)_c = (-1)^{\tilde{\omega}}\omega^{\sigma\sigma}\omega_{\alpha\beta}\omega_{\gamma\delta}(\alpha\beta\gamma\delta)_c\,.$$

Since σ is odd, $\omega^{\sigma\sigma}$ vanishes for degree reasons unless $\tilde{\omega} = 0$. When ω is even, the term in the right hand side need not vanish.

Let us assume that $\tilde{\omega} = 1$ or that $\tilde{\omega} = 0$ but $\omega^{\sigma\sigma}\omega_{\alpha\beta}\omega_{\gamma\delta}(\alpha\beta\gamma\delta)_c$ vanishes. In this case, we have $\{W_g, W_g\} = 0$ and W_g gives a marked point in the space of open string theories with underlying supermodule E, unit λ and symplectic form ω. The cyclic unital A_∞ algebra corresponding to this solution has a single product r_2^g, which is given by:

$$r_2^g(e_a, e_b) = (-1)^{\tilde{\omega}+1}\omega_{ab}\omega^{\lambda c}e_c \qquad \text{for}\, a, b \neq \sigma$$

and by the unitality constraint $-r_2^g(\lambda, e_a) = (-1)^{\tilde{a}} r_2^g(e_a, \lambda) = e_a$ for the remaining combinations of basis elements. The A_∞ constraints (2.4) reduce to:

$$r_2^g(r_2^g(x, y), z) + (-1)^{\tilde{x}} r_2^g(x, r_2^g(y, z)) = 0\,,$$

which means that $m_g := \Sigma \circ r_2 \circ \Sigma^{\otimes 2} : E^{\otimes 2} \to E$ satisfies the associativity condition:

$$m_g(m_g(x, y), z) = m_g(x, m_g(y, z))\,.$$

Moreover, the unitality constraint for r_2^g amounts to $m_g(\lambda, x) = m_g(x, \lambda) = x$. Hence the distinguished solution $W = W_g$ corresponds to an associative superalgebra structure on E, and the underlying A_∞ category reduces to an ordinary (i.e. associative) $\mathbb{Z}_2$-graded category. In the topological string theory, all integrated boundary correlators on the disk vanish and the entire information is contained in the boundary three-point functions. Fixing λ and ω, other string theories with the same units and topological metrics are given by solutions of the equations $\{W_g + W_d, W_g + W_d\} = 0$ and $\overrightarrow{\delta}_\lambda W_d = 0$, the first of which reduces to:

$$\{W_g, W_d\} + \frac{1}{2}\{W_d, W_d\} = 0\,,$$

where we used the graded antisymmetry property of the Kontsevich bracket. Letting $Q_g = \theta_{W_g}$ be the (odd) hamiltonian vector field defined by W_g (i.e. $dW_g = i_{Q_g}\omega$), this equation takes the form:

$$L_{Q_g}W_d + \frac{1}{2}\{W_d, W_d\} = 0\,. \qquad (5.22)$$

Notice that L_{Q_g} squares to zero (since $L_{Q_g}^2 = \frac{1}{2}[L_{Q_g}, L_{Q_g}] = \frac{1}{2}L_{[Q_g,Q_g]} = \frac{1}{2}L_{\theta_{\{W_g,W_g\}}} = 0$) and that it acts as an odd derivation of the Lie superalgebra $(C_R^0(A)[\tilde{\omega}], \{.,\})$ (due to the Jacobi identity for the Kontsevich bracket):

$$L_{Q_g}\{f, g\} = \{L_{Q_g}f, g\} + (-1)^{\tilde{f}+\tilde{\omega}}\{f, L_{Q_g}g\}\,.$$

Hence $(C_R^0(A)[\tilde{\omega}], L_Q, \{.,.\})$ is a differential Lie superalgebra, and (5.22) is its Maurer-Cartan equation. This means that one can study the moduli space of boundary string *theories* with fixed units and topological metrics by using the deformation theory of Lie superalgebras.

6. The noncommutative moduli space

In this section, we use the formalism developed above to construct a noncommutative version of the extended moduli space of finite D-brane systems (the boundary part of the extended moduli space of topological strings).

6.1 Symmetries

Let us fix a D-brane system described by the noncommutative generating function W, with hamiltonian derivation Q and symplectic form ω. We assume given adapted coordinates including odd loops σ_u associated with the units of the underlying A_∞ structure. It is clear from the categorical formulation of appendix A that a symmetry of the D-brane system amounts to an automorphism of the underlying cyclic and unital weak A_∞ category, called a cyclic and unital A_∞ automorphism (an automorphism is a strict autoequivalence, as appropriate for a finite category). In this subsection, we describe such symmetries as symplectomorphisms of A which obey certain supplementary properties.

Given a relative automorphism ϕ of A, we set $a' := \phi(a)$ for all $a \in \mathcal{Q}_1$, and let $V' \subset A$ be the R-sub-bimodule spanned by the elements a'. Then A is isomorphic *as a superalgebra* with the tensor algebra $T_R V'$, and ϕ can be viewed as a change of coordinates from a to a'. More precisely, the restriction of ϕ to $A_1 = V$ gives an isomorphism of R-superbimodules $\phi_1 : V \to V'$ and ϕ can be identified with the isomorphism of bigraded R-superalgebras $T_R(\phi) = \oplus_{n\geq 0}\phi^{\otimes_R n} : T_R V \to T_R V'$ induced by ϕ_1. Of course, we have $a' = \phi_1(a) = \phi(a)$ and a' is an adapted basis for the R-superbimodule V'.

R-linearity of ϕ implies that each a' is a linear combination of paths starting at $t(a)$ and ending at $h(a)$. However, ϕ need not be homogeneous with respect to the $\mathbb{N}$-grading of A, so generally each $\phi(a)$ can be a linear combination of paths of different length. Defining A'_n to be the subspace spanned by n-factor monomials in a', we have $A'_n \approx T^n_R V'$ and a new decomposition:

$$A = \oplus_{n\geq 0} A'_n$$

with $A'_0 = R$. In particular, a generic R-superalgebra automorphism induces a change of $\mathbb{N}$-grading.

Equation (4.19) implies:

$$\bar{d}\bar{\phi}(f) = \bar{\phi}^*(\bar{d}f) = \sum_a (\phi(f\overleftarrow{\delta}_a)da')_c\,,$$

which shows that the cyclic derivatives with respect to the new coordinates are given by:

$$\bar{\phi}(f)\overleftarrow{\delta}_{a'} = \phi(f\overleftarrow{\delta}_a) \Leftrightarrow \overrightarrow{\delta}_{a'}\bar{\phi}(f) = \phi(\overrightarrow{\delta}_a f)\,. \tag{6.1}$$

Relative superalgebra endomorphisms ϕ of A having the property $\phi \circ Q \circ \phi = Q$ correspond to endomorphisms of the underlying weak A_∞ category. The correspondence is obtained by expanding:

$$\phi(a) = \sum_{n\geq 0} \phi^a_{a_1\ldots a_n} a_1 \ldots a_n\,, \tag{6.2}$$

where the complex-valued coefficients $\phi^a_{a_1\dots a_n}$ vanish unless $a_1\dots a_n$ is a path starting at $t(a)$ and ending at $h(a)$. The evenness condition on ϕ gives the selection rules:

$$\phi^a_{a_1\dots a_n}=0 \qquad \text{unless}\, \tilde a=\tilde a_1+\dots+\tilde a_n\,.$$

The $n=0$ part of (6.2) stands for the sum over loops $\sum_{a\in Q_1(u,u)}\phi^a\epsilon_u$, i.e. we use the convention $\phi^a_{a_1\dots a_0}:=\phi^a\delta^u_{h(a)}\delta^u_{t(a)}\epsilon_u$. Then the A_∞ morphism associated with ϕ is given by the even R-multilinear maps $\phi_n:E[1]^n\to E[1]$ defined through:

$$\phi_n(e_{a_1}\dots e_{a_n})=\phi^a_{a_1\dots a_n}e_a\,. \tag{6.3}$$

The conditions $\phi(ab)=\phi(a)\phi(b)$ amount to the complicated relations giving the traditional definition. The maps (6.3) define an endomorphism of the weak A_∞ structure on the superbimodule E; as usual, R-multilinearity allows one to decompose them into complex-multilinear maps describing an endomorphism of the underlying weak A_∞ category. In particular, the map $\phi_0:R\to E[1]$ gives even linear maps $\phi_u:\mathbb{C}\to E_{uu}[1]$ via the decomposition $\phi_0(\sum_u\alpha_u\epsilon_u)=\sum_u\epsilon_u\phi_u(\alpha_u)\epsilon_u$ for complex α_u; these can also be viewed as the odd elements $\phi_u(1_{\mathbb{C}})=\sum_{h(a)=t(a)=u}\phi^a e_a\in E_{uu}$.

An A_∞ endomorphism of $(E,(r_n))$ is called *unital* if $\phi_1(\lambda)=\lambda$ and $\phi_n(e_{a_1}\dots e_{a_n})$ for $n\neq 1$ vanishes when any of the elements $e_{a_1}\dots e_{a_n}$ coincides with the odd A_∞ unit λ. In terms of the coefficients of ϕ, this means $\phi^a_\sigma=\delta^a_\sigma$ and $\phi^a_{a_1\dots a_n}=0$ for all $n\neq 1$, if $\sigma\in\{a_1\dots a_n\}$. Plugging this into expansion (6.2), we see that the A_∞ endomorphism is unital iff:

$$\phi(\sigma)=\sigma \qquad \text{and} \qquad \phi(a)=\text{independent of}\,\sigma \qquad \text{for all}\, a\neq\sigma\,. \tag{6.4}$$

The A_∞ endomorphism determined by ϕ is *cyclic* if $\phi^*(\omega)=\omega$; writing this condition explicitly gives a series of complicated relations used in the traditional definition. In particular, a cyclic A_∞ automorphism of $(E,\rho,(r_n)))$ amounts to a relative symplectomorphism of A preserving the homological derivation Q.

Let $\phi\in\mathrm{Aut}^\omega_{\mathrm{R}}(\mathrm{A})$ be a relative symplectomorphism. Remember from the end of subsection 3.5 that the map $\psi_\omega:C^0_R(A)[\tilde\omega]\to\mathrm{Der}^\omega_l(A)$ is equivariant with respect to the action of $\mathrm{Aut}^\omega_{\mathrm{R}}(\mathrm{A})$. Moreover, the exact sequence (4.3) shows that ψ_ω induces an isomorphism of vector spaces $C^0_R(A)[\tilde\omega]/R\approx\mathrm{Der}^\omega_l(A)$. Thus Q is ϕ-invariant iff its (canonical) hamiltonian W is invariant under the action of ϕ up to addition of elements of R:

$$\bar\phi(W)=W+\alpha \qquad \text{for some}\,\alpha\in R\,. \tag{6.5}$$

Moreover, the associated A_∞ endomorphism is unital iff ϕ satisfies relations (6.4).

Combining these observations, we see that a cyclic and unital endomorphism of the underlying A_∞ structure amounts to a symplectomorphism of A which satisfies (6.4) and (6.5). The *symmetry group* $\mathcal{G}\subset\mathrm{Aut}^\omega_{\mathrm{R}}(\mathrm{A})$ of the system is the group of all such symplectomorphisms of A. If ϕ belongs to $\mathcal{G}$, equations (6.1) and (6.4), (6.5) imply:

$$\mu=\frac{1}{2}\overrightarrow{\delta}_\sigma W=\frac{1}{2}\overrightarrow{\delta}_{\sigma'}\bar\phi(W)=\frac{1}{2}\phi(\overrightarrow{\delta}_\sigma W)=\phi(\mu)\,,$$

so ϕ preserves the moment element μ.

6.2 Algebraic construction of the noncommutative moduli space

Consider the two-sided ideal J of A generated by the elements:

$$\overrightarrow{\delta}_a W \in A \qquad (a \in \mathcal{Q}_1)\,, \tag{6.6}$$

which we shall call the *critical ideal* of the noncommutative generating function. Notice that J is also generated by $W\overleftarrow{\delta}_a$, due to the relations $\overrightarrow{\delta}_a W = (-1)^{\tilde{a}\tilde{\omega}} W\overleftarrow{\delta}_a$. We let $\mathbb{C}[\mathcal{Z}] := A/J$. Since J is $\mathbb{Z}_2$-homogeneous (being generated by $\mathbb{Z}_2$-homogeneous relations), the associative algebra $\mathbb{C}[\mathcal{Z}]$ is $\mathbb{Z}_2$ graded. Passage to $\mathbb{C}[\mathcal{Z}]$ implements the conditions:

$$\sum_{n\geq 0} \bar{W}_{aa_1\ldots a_n} a_1 \ldots a_n = 0 \qquad \forall a \in \mathcal{Q}_1 \Leftrightarrow \sum_{n\geq 0} \bar{W}_{a_1\ldots a_n a} a_1 \ldots a_n = 0 \qquad \forall a \in \mathcal{Q}_1\,,$$

which (in view of the isomorphism $V \otimes_R A \approx C^1_R(A)$) can also be written as:

$$\bar{d}W = 0\,. \tag{6.7}$$

In particular, the distinguished central element $\lambda = \sum_{u\in\mathcal{Q}_0} \lambda_u$ gives the relation:

$$\overrightarrow{\delta}_\lambda W = 0 \Leftrightarrow \mu = 0 \Leftrightarrow \omega_{ab} ab = \frac{1}{2}\omega_{ab}[a,b] = 0 \Leftrightarrow \sum_{a\in\mathcal{Q}_1} [a, a^*] = 0\,, \tag{6.8}$$

where, as usual, $[a,b] = ab - (-1)^{\tilde{a}\tilde{b}} ba$ is the supercommutator in A. Hence 'extremizing' W automatically imposes the zero-level constraint for the noncommutative moment map of [21, 22].

For $\phi \in \mathrm{Aut_R(A)}$, relations (4.19) imply $\phi^*(\bar{d}W) = (\phi(W\overleftarrow{\delta}_a)da')_c$ and $\phi^*(\bar{d}W) = \bar{d}(\bar{\phi}(W)) = (\bar{\phi}(W)\overleftarrow{\delta}_a da)_c = (\bar{\phi}(W)\overleftarrow{\delta}_a d\phi^{-1}(a'))_c$, where $a' := \phi(a)$. Expanding $d\phi^{-1}(a')$ in the second expression and comparing with the first, we find that $\phi(W\overleftarrow{\delta}_a)$ belongs to the ideal generated by $\bar{\phi}(W)\overleftarrow{\delta}_a$. This shows that relative automorphisms which preserve W (i.e. $\bar{\phi}(W) = W$) also preserve the ideal J, so they descend to R-linear automorphisms of $\mathbb{C}[\mathcal{Z}]$ (the R-superbimodule structure on $\mathbb{C}[\mathcal{Z}]$ is induced from its obvious A-superbimodule structure). In particular, the group $\mathcal{G}$ preserves J, and we obtain a group morphism $\gamma : \mathcal{G} \rightarrow \mathrm{Aut_R}(\mathbb{C}[\mathcal{Z}])$, i.e. an action of $\mathcal{G}$ by R-linear automorphisms of the superalgebra $\mathbb{C}[\mathcal{Z}]$. The canonical epimorphism $\zeta : A \rightarrow \mathbb{C}[\mathcal{Z}] = A/J$ is $\mathcal{G}$-equivariant:

$$\zeta \circ \phi = \gamma(\phi) \circ \zeta \qquad \forall \phi \in \mathcal{G}\,. \tag{6.9}$$

We set $\mathbb{C}\mathcal{M} = \mathbb{C}[\mathcal{Z}]^{\mathcal{G}}$, the homogeneous subalgebra of elements invariant under the action of $\mathcal{G}$. We will view these algebras as noncommutative coordinate rings of 'noncommutative schemes' $\mathcal{Z}$, $\mathcal{M}$, which we call the *noncommutative extended vacuum space* and *noncommutative extended moduli space* respectively.

7. The case of a single D-brane

Let us illustrate the general discussion with the simple case of a single D-brane. Then $\mathcal{Q}$ consists of m loops at a single vertex, and we let $m_\pm$ be the numbers of even and odd loops. The space of boundary observables is $E = \mathbb{C}^{m_-|m_+}$, with parity-changed dual

$V = E[1]^* = \mathbb{C}^{m_+|m_-}$. The boundary algebra R coincides with $\mathbb{C}$ while the path algebra is the free superalgebra $A = \mathbb{C}\langle\{a\}\rangle$ generated by all loops. The underlying A_∞ category has a single object, so it reduces to a weak, cyclic and unital A_∞ algebra on the supervector space E. This is the structure found in [7].

It is easy to see that $C^0(A) := C^0_{\mathbb{C}}(A)$ can be identified with the cyclic subspace $A_{\rm cyclic}$ of A, defined as the image of the idempotent operator:

$$P = \mathrm{id}_{\mathbb{C}} \oplus \oplus_{n\geq 1} \left[\frac{1}{n}\sum_{i=0}^{n-1}(\gamma_n)^i\right] \in \mathrm{End}_{\mathbb{C}}(\mathrm{A})\,.$$

Here $\gamma_n \in \mathrm{End}_{\mathbb{C}}(\mathrm{A_n})$ are the generators of the obvious $\mathbb{Z}_n$ action on A_n:

$$\gamma_n(x_1\otimes\ldots\otimes x_n) = (-1)^{\tilde{x}_1(\tilde{x}_2+\ldots+\tilde{x}_n)}x_2\otimes\ldots\otimes x_n\otimes x_1\,.$$

Thus $A_{\rm cyclic}$ consists of all polynomials $f = \sum_{n\geq 0} f_{a_1\ldots a_n}a_1\ldots a_n \in A$ whose complex coefficients satisfy the conditions $f_{a_1\ldots a_n} = (-1)^{\tilde{a}_1(\tilde{a}_2+\ldots+\tilde{a}_n)}f_{a_2\ldots a_n a_1}$. Writing $f = \sum_{n\geq 1} f_n$ with $f_n \in A_n = V^{\otimes n}$, such a polynomial belongs to $A_{\rm cyclic}$ iff $\gamma_n(f_n) = f_n$ for all n.

Let σ be the dual basis element corresponding to the parity changed A_∞ unit λ. We assume given a basis of E such that σ is one of the odd loops. The generating function is a constant-free polynomial:

$$W = \sum_{n\geq 1} W_{a_1\ldots a_n}a_1\ldots a_n \in A_{\rm cyclic}$$

in the non-commuting variables a such that $W_{a_1\ldots a_n}$ are graded-cyclic and satisfy the conditions $W_{a_1\ldots a_n} = 0$ unless $\tilde{a}_1+\cdots+\tilde{a}_n = \tilde{\omega}+1$, as well as the A_∞ constraints (5.19). Moreover, we must have $W = W_g + W_d$ with $W_g = -\frac{1}{3}\omega_{ab}[\sigma ab + (-1)^{\tilde{a}+\tilde{b}}ab\sigma + (-1)^{\tilde{b}(\tilde{a}+1)}b\sigma a]$ and where W_d vanishes at zero and is independent of σ. The matrix (ω_{ab}) satisfies properties (4.4) and (4.5) of section 4.3 and no further constraints.

Any endomorphism ϕ of A is determined by its values on the generators:

$$\phi(a) := \phi_a = \sum_{n\geq 0}\phi^a_{a_1\ldots a_n}a_1\ldots a_n$$

and can be viewed as an m-tuple $(\phi_{a_1}\ldots\phi_{a_m})$ of polynomials in the non-commuting variables a. The degree zero condition on ϕ gives the constraints $\deg\phi(a) = \tilde{a}$, so the complex coefficient $\phi^a_{a_1\ldots a_n}$ must vanish unless $\tilde{a}_1+\cdots+\tilde{a}_n = \tilde{a}$. The relative automorphism group is the usual group Aut(A) of superalgebra automorphisms. An automorphism ϕ belongs to the symmetry group $\mathcal{G}$ if $\phi(\sigma) = \sigma$, $\phi(a)$ are independent of σ for $a\neq\sigma$ and $W(\{\phi(a)\})$ equals $W(\{a\})$ as a polynomial in the non-commuting variables $\{a\}$. This imposes nonlinear algebraic conditions on the coefficients $\phi^a_{a_1\ldots a_n}$.

Observation. The automorphism group Aut(A) is a rather exotic object. In the even case $m_- = 0$, it is known[33] (see also [34]) that Aut(A) contains wild automorphisms[8] as

[8] An automorphism is called wild if it is not a composition of the so-called elementary automorphisms $(a_1\ldots a_m)\to(a_1\ldots a_i, \alpha a_i + f(a_1\ldots a_{i-1}, a_{i+1}\ldots a_m), a_{i+1}\ldots a_m)$ with $\alpha\in\mathbb{C}^*$ and f a polynomial independent of x_j.

soon as $m \geq 3$. Even in the commutative case, there are well-known open problems about automorphisms of polynomial algebras such as the jacobian conjecture. The $\mathbb{Z}_2$-graded, noncommutative case does not seem to have been studied systematically.

8. Examples

8.1 Even system with a single boundary degree of freedom

This is the simplest example relevant for topological sigma models with target spaces of even complex dimension. In this case, we have a single D-brane ($R = \mathbb{C}$) with $E = \mathbb{C}$ (concentrated in even degree), $V = \mathbb{C}[1]$ (a purely odd supervector space) and $\tilde{\omega} = 0$. The canonical forms of subsection 4.3 show that this is the only possibility when the boundary superspace has dimension one.

Up to rescaling, the boundary sector contains a single boundary observable, namely the identity operator 1, which is the even A_∞ unit. We set $\lambda = \Sigma 1$ and let σ be dual odd element in $V = \mathbb{C}[1]$ (of course, σ can be identified with λ since we identify $\mathbb{C}^*$ with $\mathbb{C}$ using the canonical basis of $\mathbb{C}$ given by the unit). The superquiver consists of the single odd loop σ, with path superalgebra $A = \mathbb{C}\langle\sigma\rangle$. By a change of normalization of the A_∞ products, we can take $\omega_{\sigma\sigma} = 1$; then $\omega_{\text{form}} = \frac{1}{2}(d\sigma^2)_c$, and σ can also be viewed as the canonical odd coordinate ξ in equation (4.10).

The non-commutative generating function is an odd polynomial $W = \sum_{n=\text{odd}} W_n(\sigma^n)_c \in C^0(A) = A_{\text{cyclic}} = A^{\text{odd}}$. The unitality constraint requires the splitting $W = W_d + W_g$ with W_d a constant. Since W must vanish at zero, this gives $W_d = 0$. Thus we must have:

$$W = W_g = -\frac{1}{3}\sigma^3$$

and the only non-trivial strict coefficient (see subsection 4.4) is $\bar{W}_{\sigma\sigma\sigma} = -1$. The A_∞ constraint (5.19) is trivially satisfied. Since W is cubic, the A_∞ algebra contains only the product r_2, which is completely determined by the unitality constraint $r_2(\lambda, \lambda) = -\lambda$. The associative product $\cdot = \Sigma \circ r_2 \circ \Sigma^{\otimes 2}$ on E is given by $1 \cdot 1 = 1$, which of course is the unique associative product on $\mathbb{C}$ with unit 1.

Since elements $\phi \in \mathcal{G}$ must preserve σ, we have $\mathcal{G} = \{\text{id}_A\}$. The critical ideal is generated by $\overrightarrow{\delta}_\sigma W = -\sigma^2$, so $J = A\sigma^2 A$ consists of all polynomials of order at least two at the origin. Thus $\mathbb{C}[\mathcal{Z}] = A/(\sigma^2)$ is the Grassmann algebra $\mathbb{C}[\sigma]$ on the odd generator σ. The noncommutative moduli space coincides with the noncommutative vacuum space, having coordinate ring:

$$\mathbb{C}[\mathcal{M}] = \mathbb{C}[\mathcal{Z}] = \mathbb{C}[\sigma]\,.$$

Hence $\mathcal{M} = \mathcal{Z} = \mathbb{C}^{0|1}$, the odd point of usual supergeometry.

In this extremely simple example, passage to the noncommutative moduli space gives nothing new, since supercommutativity is imposed as a consequence of the equations of motion.

8.2 Odd system with two boundary degrees of freedom

This is the simplest example with an odd topological boundary metric, obtained for $E = \mathbb{C}^{1|1}$. Choosing canonical coordinates as in (4.12), we have:

$$\omega_{\rm form} = (dp_0 dq_0)_c$$

with odd q_0 and even p_0, and take $\sigma = q_0$. Since σ must be a loop, the only possibility for the boundary algebra is $R = \mathbb{C}$, i.e. the index superquiver consists of an even and an odd loop at a single vertex (indeed, $\omega_{\rm form}$ vanishes for any other boundary structure). Thus $A = \mathbb{C}\langle p_0, q_0\rangle$ is a free associative superalgebra. In this case, we have $W_g = 2(q_0^2 p_0)_c$ and $W_d = W_d(p_0)$ must be a constant-free univariate polynomial in p_0. It is not hard to see that the Maurer-Cartan equation (5.22) for W_d is trivially satisfied. Indeed, it is clear that $\{W_d, W_d\} = 0$, while direct computation gives:

$$\{W_g, W_d\} = -(W_g \overleftarrow{\delta}_{q_0} \overrightarrow{\delta}_{p_0} W_d)_c = 2([q_0, p_0] \overrightarrow{\delta}_{p_0} W_d)_c = 2(q_0 [p_0, \overrightarrow{\delta}_{p_0} W_d])_c = 0\,,$$

where we noticed that the commutator $[p_0, \overrightarrow{\delta}_{p_0} W_d]$ vanishes because W_d is a polynomial in p_0. Hence the general system of this type is described by $W = W_g + W_d$, where W_d is an arbitrary constant-free polynomial in p_0. The defining equations $\overrightarrow{\delta}_a W = 0$ for the noncommutative vacuum space take the form:

$$[q_0, p_0] = 0\,, \qquad q_0^2 = -\frac{1}{2} \overrightarrow{\delta}_{p_0} W_d(p_0)\,. \tag{8.1}$$

Thus $\mathbb{C}[\mathcal{Z}] = \mathbb{C}\langle q_0, p_0\rangle / ([q_0, p_0], q_0^2 + \frac{1}{2}\overrightarrow{\delta}_{p_0} W_d(p_0))$ and $\mathcal{Z}$ is a bona-fide noncommutative superspace. It can be viewed as a 'fibration' over the noncommutative affine line $\mathbb{A}^1$ with coordinate p_0, where the (pure fuzz) fiber is a point-dependent deformation of the usual odd point $\mathbb{C}^{0|1}$.

It is known [35, 36] that all automorphisms of a free associative algebra on two generators are tame, i.e. given by iterated composition of elementary automorphisms of the form $(q_0, p_0) \to (q_0, \alpha p_0 + f(q_0))$ and $(q_0, p_0) \to (\beta q_0 + g(p_0), p_0)$ with $\alpha, \beta \in \mathbb{C}^*$ and f, g arbitrary univariate polynomials. Moreover (see [34] for a more general result), all algebra automorphisms fixing one variable are triangular. In particular, automorphisms fixing q_0 have the form $(q_0, p_0) \xrightarrow{\phi} (q_0, \alpha p_0 + f(q_0))$, with *super*algebra automorphisms obtained by restricting to even polynomials. Obviously $\phi(p_0)$ is independent of $\sigma = q_0$ iff f is the constant polynomial. In this case, ϕ is a symplectomorphism iff $\alpha = 1$. Hence $\mathcal{G}$ is a subgroup of the one-dimensional translation group:

$$\mathcal{T}: \qquad q_0 \to q_0\,, \qquad p_0 \to p_0 + t \qquad (t \in \mathbb{C})\,.$$

Such an automorphism preserves W_g up to addition of constants, and takes $W_d(p_0)$ into $W_d(p_0 + t)$. It follows that W is preserved up to constants iff W_d is linear in p_0. Thus $\mathcal{G} = \{{\rm id}_A\}$ unless $W_d = w(p_0)_c$ for some $w \in \mathbb{C}$, in which case $\mathcal{G} = \mathcal{T}$.

Hence the generic case of a nonlinear W_d gives $\mathcal{M} = \mathcal{Z}$. When $W_d = w(p_0)_c$, equations (8.1) reduce to $[q_0, p_0] = 0$ and $q_0^2 = -\frac{w}{2}$ and we find $\mathbb{C}[\mathcal{M}] = \mathbb{C}[\mathcal{Z}]^{\mathcal{T}} = \mathbb{C}\langle q_0\rangle/(q_0^2 + \frac{w}{2})$, i.e. $\mathcal{M}$ is the quantum deformation of the odd point $\mathbb{C}^{0|1}$ given by $q_0^2 = -\frac{w}{2}$.

8.3 A family of odd examples

Consider a theory with $\tilde{\omega} = 1$, where we choose adapted coordinates $p_0 \dots p_m, q_0 \dots q_m$ (with $2(m+1) = \mathrm{Card}\mathcal{Q}_0$, odd q_i and even p_i) such that that ω has canonical form (4.12). Remember from subsection 4.3 that the coordinates $a^* = \omega^{ab} b$ are given by:

$$p_i^* = q_i\,, \qquad q_i^* = -p_i\,,$$

where $*$ can be viewed as an involution on $A_1 = V$. Also remember that p_i are even and q_i are odd. We assume that $q_0 = \sigma$ corresponds to the unit.

Since $\tilde{\omega} = 1$, we have $\{W_g, W_g\} = 0$ and the discussion of subsection 5.4 applies. In particular, a general theory with the given unit and symplectic form is specified by a solution W_d of the Maurer-Cartan equation (5.22). For $\tilde{\omega} = 1$, relation (5.20) gives:

$$W_g \overleftarrow{\delta}_a = [\sigma, a^*] + \delta_a^\sigma \sum_{\alpha \in Q_1} \alpha\alpha^*$$

and the conditions $W \overleftarrow{\delta}_a = 0$ take the form:

$$[\sigma, a^*] + \delta_a^\sigma \sum_{\alpha \in Q_1} \alpha\alpha^* = -W_d \overleftarrow{\delta}_a\,,$$

where, as usual, $[.,.]$ stands for the graded commutator. In canonical coordinates, we have $[\sigma, \sigma^*] = [q_0, q_0^*] = -[q_0, p_0] = [p_0, q_0]$ and $\sum_{\alpha \in Q_1} \alpha\alpha^* = \sum_{i=0}^m [p_i, q_i]$. The noncommutative criticality constraints become:

$$2[p_0, q_0] + \sum_{i=1}^m [p_i, q_i] = -W_d \overleftarrow{\delta}_{q_0}\,, \tag{8.2}$$

and:

$$[q_0, p_i] = W_d \overleftarrow{\delta}_{q_i} \quad \forall i = 1 \dots m, \quad [q_0, q_i] = -W_d \overleftarrow{\delta}_{p_i} \qquad \forall i = 0 \dots m\,. \tag{8.3}$$

Let us assume that we are given a particular solution $W = W_g + W_d$ for which $W_d = W_d(p_1 \dots p_m)$ depends only on $p_1 \dots p_m$. Special solutions of this type were found in [27] for systems which obey $\mathbb{Z}$-valued selection rules (for the examples of [27], equation (5.22) is trivially satisfied by the ansatz $W_d = W_d(p_1 \dots p_m)$ due to the integer-valued degree condition obeyed by W in the case of Calabi-Yau compactifications). In this case, eqs. (8.2) and (8.3) reduce to the following defining relations for the noncommutative vacuum space $\mathcal{Z}$:

$$\begin{aligned}
[q_0, q_i] &= -\overrightarrow{\delta}_{p_i} W_d \qquad \forall i = 1 \dots m \\
[q_0, p_i] &= 0 \qquad \forall i = 1 \dots m \\
[q_0, p_0] &= \frac{1}{2} \sum_{j=1}^m [p_j, q_j] \\
q_0^2 &= 0\,.
\end{aligned} \tag{8.4}$$

To arrive at this form, we noticed that $\overrightarrow{\delta}_{p_i} W = W \overleftarrow{\delta}_{p_i}$ since p_i are even. Note that $[q_0, q_i] = q_0 q_i + q_i q_0$ since $[.,.]$ is the graded commutator.

For simplicity, let us consider the case when the underlying quiver has a single vertex. Then we can view $\mathcal{Z}$ as a 'fibration' over the noncommutative affine plane $\mathbb{A}^{m+1}$ with coordinates $p_0 \ldots p_m$, whose 'fiber' is a subspace of the noncommutative affine space $\mathbb{A}^{m+1}$ (with coordinates $q_0 \ldots q_m$) determined by (8.4). The abelian locus $\mathcal{Z}_{Ab}$ in $\mathcal{Z}$ is obtained by requiring that all variables supercommute. In this case, eqs. (8.4) reduce to the conditions $\partial_{p_i} W_d^{Ab}(p_1 \ldots p_m) = 0$ and we find that $\mathcal{Z}_{Ab}$ coincides with the critical locus $\mathrm{Crit}(W_d^{Ab}) \subset \mathbb{C}^m$ of W_d^{Ab}, which is the usual vacuum space expected in the supercommutative formulation. The abelianization epimorphism $\mathbb{C}\langle q_i, p_i\rangle \to \mathbb{C}[q_i, p_i]$ induces an embedding of $\mathcal{Z}_{Ab}$ into the much larger noncommutative space $\mathcal{Z}$. In the noncommutative vacuum space, one can move away from the critical locus of W_{Ab} at the price of allowing for a non-vanishing commutator of q_0 with q_i; notice that this is possible even along the locus in $\mathcal{Z}$ where p_i are required to commute. Determining the symmetry group $\mathcal{G}$ and noncommutative moduli space $\mathcal{M}$ in this class of examples is rather formidable in general and will not be attempted here. We only note that the trivial case $m = 0$ corresponds to the limit $W_d = 0$ of the example discussed in the previous subsection.

9. Conclusions

We showed that the totality of boundary tree-level data determined by a topological string theory in a finite D-brane background can be encoded *faithfully* by using the noncommutative algebraic geometry of a superquiver determined by the boundary decomposition of the D-brane system. In particular, cyclicity of integrated boundary amplitudes on the disk and the weak A_∞ constraints on such amplitudes amount to the condition $\{W, W\} = 0$, where the boundary potential W is a function defined on a *noncommutative* superspace $\mathbb{A}_Q$ determined by the quiver. We also found a differential constraint on W which expresses the presence of unit boundary observables in the boundary-preserving sector.

Fixing the bulk worldsheet data, but varying the D-brane background, gives rise to the (extended) boundary moduli space of such a system. We argued that this moduli space can be viewed as a noncommutative superspace $\mathcal{M}$ constructed as an (invariant theory) quotient of the 'noncommutative critical locus' $\mathcal{Z}$ of W by a certain group of symplectomorphisms acting on $\mathcal{Z}$.

One upshot of this analysis is that the complicated structure determined by *all* integrated boundary correlators on the disk is encoded faithfully by a form of noncommutative geometry. According to this point of view, the theory of topological D-brane deformations is intrinsically noncommutative. This gives a stringy realization of non-commutativity at the level of such moduli spaces. It should be compared with the realization of [37], which arises by translating the effective action of open strings into an action governing objects (such as connections) defined over a noncommutative space determined by the *closed* strings. In both cases, non-commutativity originates [38] from the fact that insertions of boundary observables on the disk do not commute. Hence these ostensibly different realizations are related, and it would be interesting to understand precisely how.

JHEP11(2005)032

Acknowledgments

I am grateful to V. Ginzburg for pointing out the relevance of quiver algebras, and to M. Kontsevich for sharing his ideas. This work originated from discussions during the Third Workshop on Noncommutative Algebraic Geometry held at the Mittag Leffler Institute.

A. Topological D-brane systems as cyclic and unital weak A_∞ categories

A.1 Mathematical background

A weak ($\mathbb{Z}_2$-graded) A_∞ category $\mathcal{A}$ consists of a collection of objects $Ob\mathcal{A}$ and complex supervector spaces $\mathrm{Hom}_{\mathcal{A}}(u,v) := \mathrm{Hom}(u,v)$ for $u,v \in Ob\mathcal{A}$, together with odd multilinear maps[9] $r_{u_1\ldots u_{n+1}} : \mathrm{Hom}(u_1,u_2)[1] \times \mathrm{Hom}(u_2,u_3)[1] \times \ldots \times \mathrm{Hom}(u_n,u_{n+1})[1] \rightarrow \mathrm{Hom}(u_1,u_{n+1})[1]$ for all $n \geq 0$. The case $n=0$ corresponds to odd maps $r_u : \mathbb{C} \rightarrow \mathrm{Hom}(u,u)[1]$, which amounts to giving even elements $\theta_u = r_u(1) \in \mathrm{Hom}(u,u)$. The maps r are required to satisfy certain conditions called the weak A_∞ constraints. To formulate them, let $t(x), h(x) \in Ob\mathcal{A}$ denote the tail and head of a morphism x, the unique objects of $\mathcal{A}$ such that $x \in \mathrm{Hom}(t(x),h(x))$. We say that an ordered collection of morphisms $(x_1,\ldots,x_n)$ is *composable* if $h(x_j) = t(x_{j+1})$ for all $j = 1\ldots n-1$. In this case, we let:

$$[x_1\ldots x_n] := t(x_1)t(x_2)\ldots t(x_n)h(x_n)\,, \tag{A.1}$$

viewed as a word on the set $Ob\mathcal{A}$. In particular, we set $[x] = t(x)h(x)$ for all morphisms x.

We use $|.|$ to denote the degree of homogeneous elements of $\mathrm{Hom}(u,v)$ and $\tilde{.}$ for the degree of homogeneous elements of $\mathrm{Hom}(u,v)[1]$. Then the maps r_n are required to satisfy:

$$\sum_{0\leq i+j\leq n} (-1)^{\tilde{x}_1+\ldots+\tilde{x}_i} r_{[x_1\ldots x_i][x_{i+j+1}\ldots x_n]}(x_1\ldots x_i, r_{[x_{i+1}\ldots x_{i+j}]}(x_{i+1}\ldots x_{i+j}), x_{i+j+1}\ldots x_n) = 0 \tag{A.2}$$

for any system of $\mathbb{Z}_2$-homogeneous and composable morphisms $(x_1,\ldots,x_n)$.

A weak A_∞ category is called *strong* if $r_u = 0 \Leftrightarrow \theta_u = 0$ for all $u \in Ob\mathcal{A}$ and *minimal* if it is strong and $r_{uv} = 0$ for all $u,v \in Ob\mathcal{A}$. It is called *unital* if one is given even elements $1_u \in \mathrm{Hom}(u,u)$ for each object u, such that the following conditions are satisfied:

$$\begin{aligned} r_{[x_1\ldots x_{j-1}][x_{j+1}\ldots x_n]}(x_1\ldots x_{j-1}, e_{u_j}, x_{j+1}\ldots x_n) &= 0 \qquad \text{for all}\, n \neq 2 \quad \textit{andall}\, j \\ r_{[\lambda_u,x]}(\lambda_u,x) = -x\,, \quad r_{[y,\lambda_u]}(y,\lambda_u) &= (-1)^{\tilde{y}} y\,, \end{aligned} \tag{A.3}$$

where $\lambda_u := \Sigma 1_u \in \mathrm{Hom}(u,u)[1]$ and $u_j := h(x_{j-1}) = t(x_{j+1})$. In these relations, it is understood that $(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)$ is any composable system consisting of $\mathbb{Z}_2$-homogeneous elements. The last conditions in (A.3) are imposed for any homogeneous $x \in \mathrm{Hom}(u,v)$ and $y \in \mathrm{Hom}(v,u)$, with arbitrary u,v. The elements $1_u, \lambda_u$ are called *units* and *odd units* respectively.

[9] Notice that we take morphisms to compose *forward* under the A_∞ products.

The A_∞ category is called *cyclic* if one is given non-degenerate[10] bilinear forms ρ_{uv} : $\mathrm{Hom}(u,v)\times \mathrm{Hom}(v,u)\to \mathbb{C}$, homogeneous of the same $\mathbb{Z}_2$-degree $\tilde{\omega}$, such that $\rho_{uv}(x,y) = (-1)^{|x||y|}\rho_{vu}(y,x)$ and such that the following identities are satisfied:

$$\rho_{t(x_0)h(x_0)}(x_0, r_{[x_1\dots x_n]}(x_1\dots x_n)) = (-1)^{\tilde{x}_0(\tilde{x}_1+\dots+\tilde{x}_n)}\rho_{t(x_1)h(x_1)}(x_1, r_{[x_2\dots x_0]}(x_2\dots x_n, x_0))\,, \tag{A.4}$$

whenever $(x_0, x_1, \dots, x_n)$ is a homogeneous composable system and $[x_0\dots x_n]$ is a cyclic word, i.e. $h(x_n) = t(x_0)$. Equivalently,

$$\begin{aligned}&\omega_{t(x_0)h(x_0)}(x_0, r_{[x_1\dots x_n]}(x_1\dots x_n)) = \\ &\quad = (-1)^{\tilde{x}_0+\tilde{x}_1+\tilde{x}_0(\tilde{x}_1+\dots+\tilde{x}_n)}\omega_{t(x_1)h(x_1)}(x_1, r_{[x_2\dots x_0]}(x_2\dots x_n, x_0))\,.\end{aligned} \tag{A.5}$$

where $\omega_{uv} = \rho_{uv}\circ\Sigma^{\otimes 2}$ (i.e. $\omega_{uv}(x,y) = (-1)^{\tilde{x}}\rho_{uv}(x,y)$) are the suspended bilinear forms $\omega_{uv} : \mathrm{Hom}(u,v)[1]\times \mathrm{Hom}(v,u)[1]\to\mathbb{C}$, which satisfy $\omega_{uv}(x,y) = (-1)^{\tilde{x}\tilde{y}+1}\omega_{vu}(y,x)$.

The concept of A_∞ category was introduced in [48] as a generalization of the notion of A_∞ algebra [55]. These objects are studied mathematically in [49, 41, 42, 50−54]. It is by now well-understood that they play an important role in homological algebra, in particular in giving a natural formulation of derived categories [56]. They also play a crucial role in the homological mirror symmetry program [17].

A.2 Category-theoretic description of topological D-brane systems

It was pointed out in [8] (and derived in [7] from the worldsheet perspective, see also [9, 10]) that topological D-branes in string theory form the structure of a weak, cyclic and unital A_∞ category. In this realization, the D-branes of the theory are the objects of A, while each morphism space $\mathrm{Hom}_{\mathcal{A}}(u,v)$ is the space of zero-form topological observables for a string stretching from u to v. The bilinear forms ρ_{uv} are the topological metrics, while the A_∞ units 1_u are the identity observables in the boundary sectors $\mathrm{Hom}_{\mathcal{A}}(u,u)$. The A_∞ products arise by dualizing the integrated correlators on the disk. These can be recovered from the former with the aid of the topological metrics:

$$\langle\langle x_0\dots x_n\rangle\rangle = \rho_{h(x_0)t(x_0)}(x_0, r_{[x_1\dots x_n]}(x_1\dots x_n))\,,$$

for composable $(x_0\dots x_n)$ such that $[x_0\dots x_n]$ is a cyclic word. We refer the reader to [7, 8] for further details.

As explained in [7, 8], non-vanishing maps r_u are present when the background of the topological string theory does not satisfy the string equations of motion. In this case, the elements $\theta_u = r_u(1)\in \mathrm{Hom}_{\mathcal{A}}(u,u)$ correspond to tadpoles. This can usually be corrected by shifting the string vacuum until all tadpoles are eliminated [8, 7]. One sometimes has an obstruction to reaching a solution, a phenomenon which was originally noticed in [57].

The structure described above is a homotopy-theoretic generalization of the boundary data of a topological field theory defined on open Riemann surfaces, which was studied in [23, 39, 40]. The latter data arises by forgetting all integrated boundary correlators,

[10]This means that all linear maps $\mathrm{Hom}(v,u)\to\mathrm{Hom}(u,v)^*$ determined by ρ_{uv} are bijective.

JHEP11(2005)032

thereby keeping only the information contained in the boundary units, topological metrics and three-point functions.

The fact that weak cyclic and unital A_∞ categories describe topological D-brane systems is implicit in the homological mirror symmetry conjecture [17] and in the work of Fukaya and collaborators [48, 57, 41]. The physical interpretation of this was discussed in [8], the dG case having been considered previously in [28, 29, 58]. A connection with the D-brane superpotential and Chern-Simons theory was made in [43, 8], realizing explicitly an observation originally due to [60]. See [59, 44 – 46] for further physics discussion. A_∞ algebras as descriptions of open string vertices appeared originally in [13] in the context of bosonic string field theory, which was further studied in [12]. As discussed in [43, 8, 12], the A_∞ structure describing open topological strings can also be obtained from the results of [13], by using the well-known formal analogy of bosonic and topological string theories.

B. Some basic isomorphisms

Convention (2.1) for the superbimodule structure of the dual affects some standard isomorphisms. Given two R-superbimodules U, V, the supervector space $\mathrm{Hom}_{\mathrm{Mod-R}}(U^{\mathrm{opp}}, V)$ becomes an R- superbimodule with respect to the external multiplications given by $(\alpha\phi\beta)(x)$ $= \alpha\phi(x\beta)$. There exists an isomorphism of R-superbimodules $\mathrm{Hom}_{\mathrm{Mod-R}}(U^{\mathrm{opp}}, V) \approx V \otimes_R U^{\mathrm{v}}$, whose inverse takes $y \otimes_R f \in V \otimes_R U^{\mathrm{v}}$ into the right R-supermodule morphism $\phi : U^{\mathrm{opp}} \to V$ given by $\phi(x) := yf(x)$. This isomorphism maps the R-sub-bimodule $\mathrm{Hom}_{\mathrm{R-Mod-R}}(U^{\mathrm{opp}}, V)$ of $\mathrm{Hom}_{\mathrm{Mod-R}}(U^{\mathrm{opp}}, V)$ into the center of $V \otimes_R U^{\mathrm{v}}$:

$$\underline{\mathrm{Hom}}(U^{\mathrm{opp}}, V) = \mathrm{Hom}_{\mathrm{R-Mod-R}}(U^{\mathrm{opp}}, V) \approx [V \otimes_R U^{\mathrm{v}}]^R . \tag{B.1}$$

In particular, we have $\underline{\mathrm{Hom}}(U^{\mathrm{opp}}, U) \approx [U^{\mathrm{v}} \otimes_R U^{\mathrm{v}}]^R$, so given a bilinear form σ on U, the map $j_\sigma : U^{\mathrm{opp}} \to U^{\mathrm{v}}$ defined in section 2 can be identified with a central element $\hat{\sigma} \in [U^{\mathrm{v}} \otimes_R U^{\mathrm{v}}]^R$. Tracing through the identifications, one finds that σ can be recovered from $\hat{\sigma}$ as follows. If $\hat{\sigma} = \sum_i f_i \otimes_R g_i$ with $f_i, g_i \in U^{\mathrm{v}}$, then $\sigma(x, y) = \sum_i f_i(xg_i(y))$.

C. Explicit construction of the differential envelope

The construction below is a slight adaptation of that given [47]. Let A be an R-superalgebra.

Definition 4. *Consider the R-superbimodule $A_R := A/R$ and let $\nu : A \to A_R$ be the natural projection. We let $\nu(a) := \bar{a}$ for all $a \in A$ and let $K_R = A \oplus A_R$, viewed as an R-superbimodule. Define:*

$$\Omega_R A = T_R K_R / J \, , \tag{C.1}$$

where $T_R K$ (endowed with its obvious $\mathbb{Z} \times \mathbb{Z}_2$ grading) is the tensor algebra on K and $J \subset T_R K_R$ is the $\mathbb{Z} \times \mathbb{Z}_2$-homogeneous two-sided ideal generated by all elements of the form $\overline{a} \otimes b + a \otimes \overline{b} - \overline{ab}$ and $a \otimes b - ab$ with $a, b \in A$. We let d be the unique R-linear derivation *of $\Omega_R A$ of bidegree $(1,0) \in \mathbb{Z} \times \mathbb{Z}_2$ (with respect to the pairing (1.4)) which satisfies the relations:*

$$da = \bar{a}\,, \qquad d\bar{a} = 0 \qquad \textit{for all}\, a \in A \tag{C.2}$$

(this derivation is well-defined). The $\mathbb{Z} \times \mathbb{Z}_2$-grading of $\Omega_R A$ is induced by the obvious $\mathbb{Z} \times \mathbb{Z}_2$-grading of $T_R K_R$.

If $\cdot$ denotes multiplication in $\Omega_R A$, we find that $\Omega_R A$ is spanned by all finite products of elements of the form a and $db = \bar{b}$ with $a, b \in A$, with the relations $d(ab) = (da)\cdot b + a\cdot(db)$ and $a \cdot b = ab$ Notice that the unit element of $\Omega_R A$ coincides with 1_A due to the relation $1_A \cdot a = 1_A a = a = a 1_A = a \cdot 1_A$ for $a \in A$, and that d squares to zero due to the relations $d\bar{a} = 0 \Leftrightarrow d^2 a = 0$ for $a \in A$. Moreover, any element of Ω_A can be written as a finite sum of elements of the form $a_0 \cdot da_1 \cdot \ldots \cdot da_n$ with $a_i \in A$, by applying the identity:

$$\begin{aligned}
&a_0 \cdot \bar{a}_1 \cdot \ldots \cdot \bar{a}_n \cdot b_0 \cdot \bar{b}_1 \cdot \ldots \cdot \bar{b}_m := \\
&\quad := a_0 \cdot \bar{a}_1 \cdot \ldots \bar{a}_{n-1} \cdot \overline{a_n b_0} \cdot \ \bar{b}_1 \cdot \ldots \cdot \bar{b}_m + \\
&\qquad + \sum_{i=1}^{n-1} (-1)^{n-i} a_0 \cdot \bar{a}_1 \cdot \ldots \cdot \overline{a_{n-i} a_{n-i+1}} \cdot \ldots \cdot \bar{a}_n \cdot \bar{b}_0 \cdot \ldots \cdot \bar{b}_m + \\
&\qquad + (-1)^n (a_0 a_1) \cdot \bar{a}_2 \cdot \ldots \cdot \bar{a}_n \cdot \bar{b}_0 \cdot \ldots \cdot \bar{b}_m \\
&\text{for all} \ \ a_0 \ldots a_n, b_0 \ldots b_m \in A\,,
\end{aligned}$$

which follows from the relations valid in $\Omega_R A$. Furthermore, we can denote the product in $\Omega_R A$ by juxtaposition.

D. Coefficient expression for the cyclic bracket

Proposition 8. *Let $f, g \in C^0_R(A)$ have the forms:*

$$\begin{aligned}
f &= \sum_{n\geq 0} f_{a_1\ldots a_n} (a_1 \ldots a_n)_c = c(f) + \sum_{n \geq 1} \frac{\bar{f}_{a_1\ldots a_n}}{n} (a_1 \ldots a_n)_c \\
g &= \sum_{n\geq 0} g_{a_1\ldots a_n} (a_1 \ldots a_n)_c = c(g) + \sum_{n \geq 1} \frac{\bar{g}_{a_1\ldots a_n}}{n} (a_1 \ldots a_n)_c
\end{aligned}$$

with strict coefficients $\bar{f}_{a_1\ldots a_n}$, $\bar{g}_{a_1\ldots a_n}$ and assume that both f and g are homogeneous of degree $\tilde{\omega} + 1 \in \mathbb{Z}_2$. Then the cyclic bracket of f and g:

$$\{f, g\} = c(\{f,g\}) + \sum_{n\geq 1} \frac{\bar{\phi}_{a_1\ldots a_n}}{n} (a_1 \ldots a_n)_c\,, \tag{D.1}$$

has strict coefficients:

$$\begin{aligned}
\bar{\phi}_{a_1\ldots a_n} \ = \sum_{j=0}^{n} \sum_{i=0}^{n-j} (-1)^{\tilde{a}_1 + \ldots + \tilde{a}_i} [&\bar{f}_{a_1 \ldots a_i a a_{i+j+1} \ldots a_n} \bar{g}^a{}_{a_{i+1}\ldots a_{i+j}} + \\
&+ \bar{g}_{a_1 \ldots a_i a a_{i+j+1} \ldots a_n} \bar{f}^a{}_{a_{i+1}\ldots a_{i+j}}]\,.
\end{aligned} \tag{D.2}$$

and:

$$c(\{f,g\}) := \bar{f}_a \bar{g}^a = \bar{f}^a \bar{g}_a \,, \tag{D.3}$$

where we lift coefficients with ρ^{ab} as in (4.32).

With our conventions, expression (D.3) can be obtained by formally setting $n = 0$ in equation (D.2)) and dividing by two. The second equality in (D.3) follows upon using the symmetry property of ρ^{ab} and the selection rule $\tilde{a} = \tilde{\omega} + 1$. *Proof.* By proposition 6, we have:

$$\{f,g\} = \sum_{n\geq 0}\sum_{j=0}^{n} F^{(j)}_{(a_1\dots a_n)}(a_1\dots a_n)_c \tag{D.4}$$

where:

$$F^{(j)}_{a_1\dots a_n} = \bar{f}_{a_1\dots a_j b}\omega^{ba}\bar{g}_{a a_{j+1}\dots a_n} \,. \tag{D.5}$$

The case $n = 0$ can be checked directly, so we discuss only the case $n \geq 1$. If $i + j \leq n$, we compute:

$$F^{(j)}_{a_{i+1}\dots a_n, a_1\dots a_i} = \bar{f}_{a_{i+1}\dots a_{i+j}b}\omega^{ba}\bar{g}_{a a_{i+j+1}\dots a_n a_1\dots a_i} = (-1)^{\sigma_1}\bar{g}_{a_1\dots a_i a a_{i+j+1}\dots a_n}\bar{f}^a{}_{a_{i+1}\dots a_{i+j}} \,,$$

where we used cyclicity of f and g to permute indices. The sign factor is easily determined keeping in mind the symmetry property (4.8) and the selection rule (4.9):

$$\sigma_1 = \tilde{\omega}(\tilde{a}+1) + (\tilde{a}_1 + \ldots + \tilde{a}_i)(\tilde{a} + \tilde{a}_{i+j+1} + \ldots + \tilde{a}_n) + (\tilde{a}+\tilde{\omega})(\tilde{a}_{i+1} + \ldots + \tilde{a}_{i+j}) \pmod 2 \,.$$

For $i + j > n$, we find:

$$\begin{aligned} F^{(j)}_{a_{i+1}\dots a_n, a_1\dots a_i} &= (-1)^{\tilde{a}+\tilde{\omega}+1}\bar{f}_{a_{i+1}\dots a_n a_1\dots a_{j-n+i} a}\bar{g}^a{}_{a_{j-n+i+1}\dots a_i} \\ &= (-1)^{\sigma_2}\bar{f}_{a_1\dots a_{j-n+i} a a_{i+1}\dots a_n}\bar{g}^a{}_{a_{j-n+i+1}\dots a_i} \,, \end{aligned}$$

with:

$$\sigma_2 = \tilde{a} + \tilde{\omega} + 1 + (\tilde{a}_{i+1} + \ldots + \tilde{a}_n)(\tilde{a} + \tilde{a}_1 + \ldots + \tilde{a}_{j-n+i}) \pmod 2 \,.$$

This allows us to write:

$$\sum_{j=0}^{n} F^{(j)}_{(a_1\dots a_n)} = \frac{1}{n}\left[\sum_{j=0}^{n}\sum_{i=1}^{n-j}(-1)^{\epsilon_1}F^{(j)}_{a_{i+1}\dots a_n, a_1\dots a_i} + \sum_{j=0}^{n}\sum_{i=n-j+1}^{n}(-1)^{\epsilon_2}F^{(j)}_{a_{i+1}\dots a_n, a_1\dots a_i}\right] \tag{D.6}$$

where:

$$\begin{aligned} \epsilon_1 &= \tilde{a} + \tilde{\omega} + (\tilde{a} + \tilde{\omega} + \tilde{a}_1 + \ldots + \tilde{a}_i)(\tilde{a} + \tilde{a}_{i+1} + \ldots + \tilde{a}_{i+j}) \\ \epsilon_2 &= 1 + \tilde{a} + \tilde{\omega} + (\tilde{a}_{i+1} + \ldots + \tilde{a}_n)(\tilde{a} + \tilde{a}_{j-n+i+1} + \ldots + \tilde{a}_i) \,. \end{aligned}$$

We now perform the replacement $i \to i' = i + j - n, j \to j' = n - j$ in the second double sum. Denoting the new summation indices i', j' by i, j, this gives:

$$\begin{aligned} &\sum_{j=0}^{n} F^{(j)}_{(a_1\dots a_n)} = \\ &\quad = \frac{1}{n}\sum_{j=0}^{n}\sum_{i=1}^{n-j}\left[(-1)^{\epsilon_1}\bar{g}_{a_1\dots a_i a a_{i+j+1}\dots a_n}\bar{f}^a{}_{a_{i+1}\dots a_{i+j}} + (-1)^{\tilde{\epsilon}_2}\bar{f}_{a_1\dots a_i a a_{i+j+1}\dots a_n}\bar{g}^a{}_{a_{i+1}\dots a_{i+j}}\right] \,, \end{aligned}$$

where:

$$\tilde{\epsilon}_2 = 1 + \tilde{a} + \tilde{\omega} + (\tilde{a} + \tilde{a}_{i+1} + \cdots + \tilde{a}_{i+j})(\tilde{a}_{i+j+1} + \cdots + \tilde{a}_n)\,.$$

The next step is to notice that the first term in square brackets vanishes unless:

$$\tilde{a} + \tilde{a}_1 + \cdots + \tilde{a}_i = \tilde{g} + \tilde{a}_{i+j+1} + \cdots + \tilde{a}_n \pmod 2$$

and:

$$\tilde{a} + \tilde{a}_{i+1} + \ldots + \tilde{a}_{i+j} = \tilde{f} + \tilde{\omega} \pmod 2\,,$$

which allows us to replace ϵ_1 by:

$$\epsilon_1' = \tilde{a} + \tilde{\omega} + (\tilde{f} + \tilde{\omega})(\tilde{g} + \tilde{\omega} + \tilde{a}_{i+j+1} + \cdots + \tilde{a}_n)\,.$$

Similarly, the second term in the square brackets vanishes unless:

$$\tilde{a} + \tilde{a}_{i+1} + \ldots + \tilde{a}_{i+j} = \tilde{g} + \tilde{\omega}\,,$$

which allows us to replace $\tilde{\epsilon}_2$ by:

$$\epsilon_2' = \tilde{a} + \tilde{\omega} + 1 + (\tilde{g} + \tilde{\omega})(\tilde{a}_{i+j+1} + \cdots + \tilde{a}_n)\,.$$

Finally, using the assumption $\tilde{f} = \tilde{g} = \tilde{\omega} + 1$, we find:

$$\epsilon_1' = \epsilon_2' = 1 + \tilde{\omega} + \tilde{a} + \tilde{a}_{i+j+1} + \ldots + \tilde{a}_n = \tilde{a}_1 + \ldots + \tilde{a}_i \pmod 2\,,$$

where we used the selection rule $\tilde{a}_1 + \cdots + \tilde{a}_i + \tilde{a} + \tilde{a}_{i+j+1} + \cdots + \tilde{a}_n = \tilde{f} = \tilde{g} = \tilde{\omega} + 1$.

Corollary 2. *Let $f = c(f) + \sum_{n\geq 0} \frac{\bar{f}_{a_1\ldots a_n}}{n}(a_1 \ldots a_n)_c$ be a cyclic element of A of degree $\tilde{\omega} + 1$. Then:*

$$\frac{1}{2}\{f,f\} = \frac{1}{2}\bar{f}_a\bar{f}^a + \sum_{n\geq 1}\frac{1}{n}\left(\sum_{0\leq i+j\leq n}(-1)^{\tilde{a}_1+\ldots+\tilde{a}_i}\bar{f}_{a_1\ldots a_i a a_{i+j+1}\ldots a_n}\bar{f}^a{}_{a_{i+1}\ldots a_{i+j}}\right)(a_1\ldots a_n)_c\,.$$

References

[1] E. Witten, *Mirror manifolds and topological field theory*, `hep-th/9112056`.

[2] B. Dubrovin, *Geometry of 2D topological field theories*, `hep-th/9407018`.

[3] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *Topological strings in $D < 1$*, *Nucl. Phys.* **B 352** (1991) 59.

[4] S. Barannikov, M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, *Internat. Math. Res. Notices* **4** (1998) 201-215, [`alg-geom/9710032`].

[5] G. Tian , *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, in *Mathematical aspects of string theory,* 629-646, S.T. Yau ed., World Scientific, Singapore, 1987.

[6] A. Todorov, *The Weil-Petersson geometry of the moduli space of* SU$(n \geq 3)$ *(Calabi-Yau) manifolds, I*, *Commun. Math. Phys.* **126** (1989) 325.

[7] M. Herbst, C.-I. Lazaroiu and W. Lerche, *Superpotentials, A_∞ relations and wdvv equations for open topological strings*, *JHEP* **02** (2005) 071 [hep-th/0402110].

[8] C.I. Lazaroiu, *D-brane categories*, *Int. J. Mod. Phys.* **A 18** (2003) 5299 [hep-th/0305095].

[9] K. Costello, *The A-infinity operad and the moduli space of curves*, math.AG/0402015.

[10] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, math.QA/0412149.

[11] H. Kajiura, J. Stasheff, *Homotopy algebras inspired by classical open-closed string field theory*, math.QA/0410291

[12] H. Kajiura, *Noncommutative homotopy algebras associated with open strings*, Ph.D. thesis, math.QA/0306332.

[13] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories, I. Foundations*, *Nucl. Phys.* **B 505** (1997) 569 [hep-th/9705038].

[14] M. Kontsevich, *Formal (non)commutative symplectic geometry*, *The Gelfand mathematical seminars, 1990–1992*, 173–187, Birkhäuser Boston, Boston, MA, 1993.

[15] L. LeBruyn, *Noncommutative geometry @n*, unpublished book, available at www.neverendingbooks.org;
see also *Three talks on noncommutative geometry*, math.RA/0312221.

[16] V. Ginzburg, *Lectures on noncommutative geometry*, math.AG/0506603

[17] M. Kontsevich, *Homological algebra of mirror symmetry*, *Proc. Internat. Congr. Math.*, Zürich, Switzerland 1994, Basel, vol. 1, Birkhäuser Verlag, 1995, 120–139.

[18] A. Polishchuk, *Homological mirror symmetry with higher products*, *Winter school on mirror symmetry, vector bundles and lagrangian submanifolds*, Cambridge, Massachusetts, 1999 in AMS/IP Stud. Adv. Math. **23** pp. 247–259 [math.AG/9901025].

[19] G.-C. Rota, B. Sagan, P. R. Stein, *A cyclic derivative in noncommutative algebra*, *J. of Algebra* **64** (1980) 54-75.

[20] D. Voiculescu, *A note on cyclic gradients*, *Indiana Univ. Math. J* **49** (2000) 3 837-841.

[21] W. Crawley-Boevey, P. Etingof, V. Ginzburg, *Noncommutative geometry and quiver algebras*, math.AG/0502301.

[22] M. Van den Bergh, *Double Poisson algebras*, math.QA/0410528.

[23] C.I. Lazaroiu, *On the structure of open-closed topological field theory in two dimensions*, *Nucl. Phys.* **B 603** (2001) 497 [hep-th/0010269].

[24] J. Cuntz, D. Quillen, *Algebra extensions and nonsingularity*, *Journal AMS* **8** (1995) 251-289.

[25] V. Ginzburg, *Noncommutative symplectic geometry, quiver varieties and operads*, *Math. Res. Lett* **8** (2001) 3, p. 377.

[26] R. Bocklandt, L. Le Bruyn, *Necklace Lie algebras and noncommutative symplectic geometry*, *Math. Z.***240** (2002) 141-167 [math.AG/0010030].

[27] P.S. Aspinwall and S. Katz, *Computation of superpotentials for D-branes*, hep-th/0412209.

[28] C.I. Lazaroiu, *Generalized complexes and string field theory*, *JHEP* **06** (2001) 052 [hep-th/0102122].

[29] C.I. Lazaroiu, *Unitarity, D-brane dynamics and D-brane categories*, *JHEP* **12** (2001) 031 [hep-th/0102183].

[30] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory*, *Int. J. Mod. Phys.* **A 12** (1997) 1405 [hep-th/9502010].

[31] M. Penkava, *Infinity algebras, cohomology and cyclic cohomology, and infinitesimal deformations*, math.QA/0111088.

[32] A. Hamilton, A. Lazarev, *Homotopy algebras and noncommutative geometry*, math.QA/0410621.

[33] U.U. Umirbaev, *Tame and wild automorphisms of polynomial algebras and free associative algebras*, Max-Planck Institute Bonn, Preprint MPIM2004-108.

[34] V. Drensky, J.T. Yu, *The strong Anick conjecture is true*, math.RA/0507170.

[35] A.J. Czerniakiewicz, *Automorphisms of a free associative algebra of rank 2, I Trans. Amer. Math. Soc.* **160** (1971) 393; *Automorphisms of a free associative algebra of rank 2, II Trans. Amer. Math. Soc.* **171** (1972) 309.

[36] L.G. Makar-Limanov, *On automorphisms of free algebra with two generators*, *Functional Anal. Appl.***4** (1970) 262-263.

[37] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[38] V. Schomerus, *D-branes and deformation quantization*, *JHEP* **06** (1999) 030 [hep-th/9903205].

[39] G. Moore and G. Segal, unpublished; see http://online.kitp.ucsb.edu/online/mp01/.

[40] G.W. Moore, *Some comments on branes, G-flux and K-theory*, *Int. J. Mod. Phys.* **A 16** (2001) 936 [hep-th/0012007].

[41] K. Fukaya, *Floer homology and mirror symmetry, II*, in *Minimal surfaces, geometric analysis and symplectic geometry*, Baltimore, MD, 1999, Adv. Stud. Pure Math. **34**, Math. Soc. Japan, Tokyo, 2002, pp. 31–127.

[42] M. Kontsevich, Y. Soibelman, *Homological mirror symmetry and torus fibrations*, math.SG/0011041.

[43] C.I. Lazaroiu, *String field theory and brane superpotentials*, *JHEP* **10** (2001) 018 [hep-th/0107162].

[44] C.I. Lazaroiu and R. Roiban, *Holomorphic potentials for graded D-branes*, *JHEP* **02** (2002) 038 [hep-th/0110288].

[45] C.I. Lazaroiu, *An analytic torsion for graded D-branes*, *JHEP* **09** (2002) 023 [hep-th/0111239].

[46] C.I. Lazaroiu and R. Roiban, *Gauge-fixing, semiclassical approximation and potentials for graded Chern-Simons theories*, *JHEP* **03** (2002) 022 [hep-th/0112029].

[47] R. Coquereaux, D. Kastler, *Remarks on the differential envelopes of associative algebras*, *Pacific. J. Math* (137) (1989) 2.

[48] K. Fukaya, *Morse homotopy, A_∞-category, and Floer homologies*, Proc. of GARC Workshop on Geometry and Topology '93, H.J. Kim, ed., Lecture Notes **18**, Seoul Nat. Univ., Seoul, 1993, pp. 1–102.

[49] B. Keller, *Introduction to A_∞ algebras and modules*, Homology, Homotopy and Applications **3** (2001)1, pp 1–35 [`math.RA/9910179`].

[50] K. Lefvre-Hasegawa, *Sur les A-infini catégories*, Ph.D. Thesis, `math.CT/0310337`

[51] V. V. Lyubashenko, *Category of A_∞-categories*, *Homology, Homotopy and Applications* **5** (2003) 1,pp. 1–48 [`math.CT/0210047`].

[52] V. Lyubashenko, S. Ovsienko, *A construction of an A-infinity category*, `math.CT/0211037`

[53] V. Lyubashenko, O. Manzyuk, *Quotients of unital A_∞-categories*, `math.CT/0306018`

[54] V. Lyubashenko, O. Manzyuk, *Free A_∞-categories*, `math.CT/0312339`.

[55] J. D. Stasheff, *Homotopy associativity of H-spaces, I & II*, *Trans. Amer. Math. Soc.* **108** (1963), p. 275.

[56] M. Kontsevich, Y. Soibelman, book in preparation

[57] K. Fukaya, Y.G. Oh, H. Ohta, K. Ono, *Lagrangin intersection Floer theory — anomaly and obstructon*, preprint.

[58] C.I. Lazaroiu, *Graded lagrangians, exotic topological D-branes and enhanced triangulated categories*, *JHEP* **06** (2001) 064 [`hep-th/0105063`].

[59] C.I. Lazaroiu, R. Roiban and D. Vaman, *Graded Chern-Simons field theory and graded topological D-branes*, *JHEP* **04** (2002) 023 [`hep-th/0107063`].

[60] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637–678 [`hep-th/9207094`].

[61] V. Ginzburg, *Double derivations and Cyclic homology*, `math.KT/0505236`.

JHEP11(2005)032